Classification of Covering Spaces and Canonical Change of Basepoint

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*Formalized in Coq using HoTT library

https://gitlab.tue.nl/computer-verified-proofs/covering-spaces

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Motivation

Homotopical interpretation

- $X : \mathbf{Type}$ is a space
- $p : a =_X b$ is a path
- Paths between paths $p =_{(a =_X b)} q$

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How do you **formalize** results from Algebraic Topology into HoTT?

Can we get some **intuition**?

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- Degree
- Covering spaces
- Homology

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No success, but formalized some parts of algebraic topology

- Classification of Covering Spaces
- Canonical Change of Basepoint

Classification of Covering Spaces

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*already shown in [Buchholtz, Van Doorn, Rijke (2018)]

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- instead of $p : \tilde{X} \to X$ work directly with fibers $F(x) :\equiv p^{-1}(x)$
  ↪ easier, replace propositional (=) by judgmental (≡) equalities
- no mention of continuity / local, point-set topological properties
  ↪ automatic! 'bad' constructions impossible in HoTT

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A posteriori justification

- Right notion of equality $F_1 = F_2$ implies $h : \prod_{x:X} F_1(X) \simeq F_2(X)$
- Prove classical theorems!

Classification

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For a connected, pointed type (X, x_0)

F ⟷ H

F: pointed, connected covering space of X

H: subgroup $\pi_1(X, x_0)$

i.e. predicate $\pi_1(X, x_0) \to \mathbf{Prop}$ closed under group operations

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$$F \longleftrightarrow H$$

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From covering space to subgroup

$F \quad \longmapsto \quad H_F,$ loops p in X for which there exists a loop in the covering space lying over p

- **Surjective** via the **universal covering space**
- **Injective** via the **lifting criterion**

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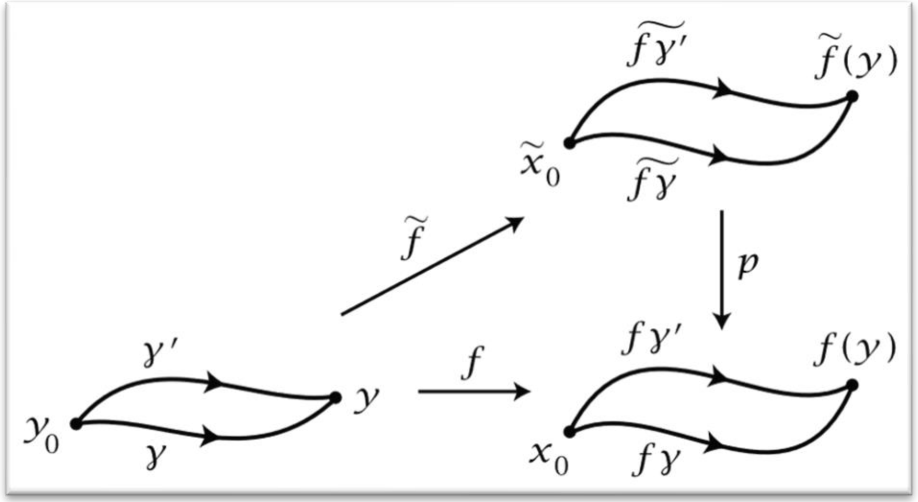
Lifting Criterion in HoTT

Hatcher

Proposition 1.33. *Suppose given a covering space* $p:(\widetilde{X},\tilde{x}_0)\to(X,x_0)$ *and a map* $f:(Y,y_0)\to(X,x_0)$ *with* Y *path-connected and locally path-connected. Then a lift* $\tilde{f}:(Y,y_0)\to(\widetilde{X},\tilde{x}_0)$ *of* f *exists iff* $f_*(\pi_1(Y,y_0))\subset p_*(\pi_1(\widetilde{X},\tilde{x}_0))$.



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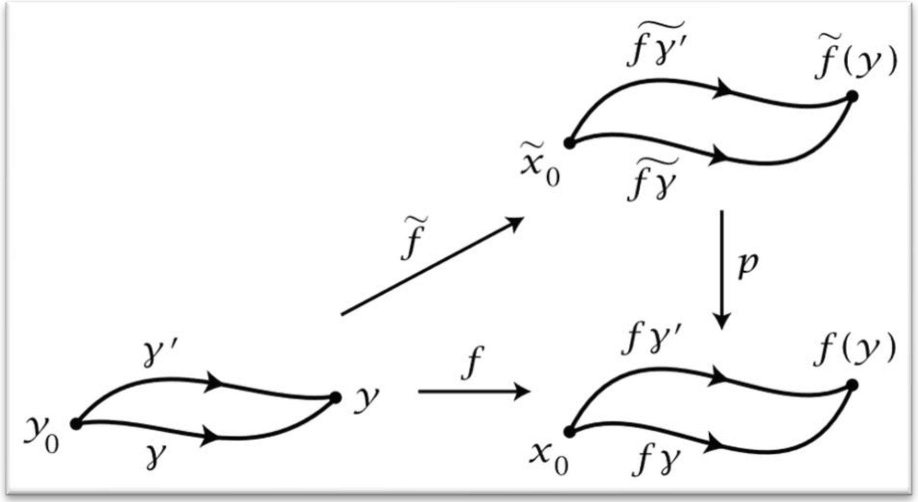


Definitions needed in HoTT

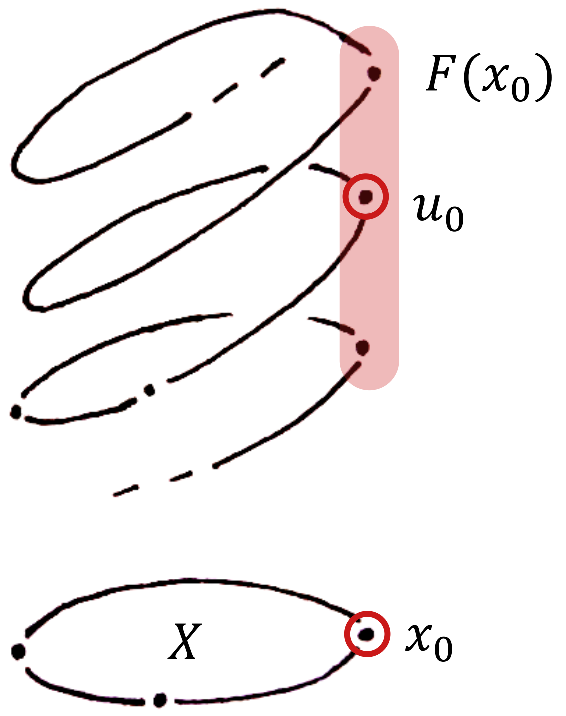
- pointed covering space
- total space and the covering map
- lift of a pointed map to the covering space

Definitions in HoTT

- **Pointed covering space** of (X, x_0)

  family $F : X \to \mathbf{Set}$

  with a point $u_0 : F(x_0)$



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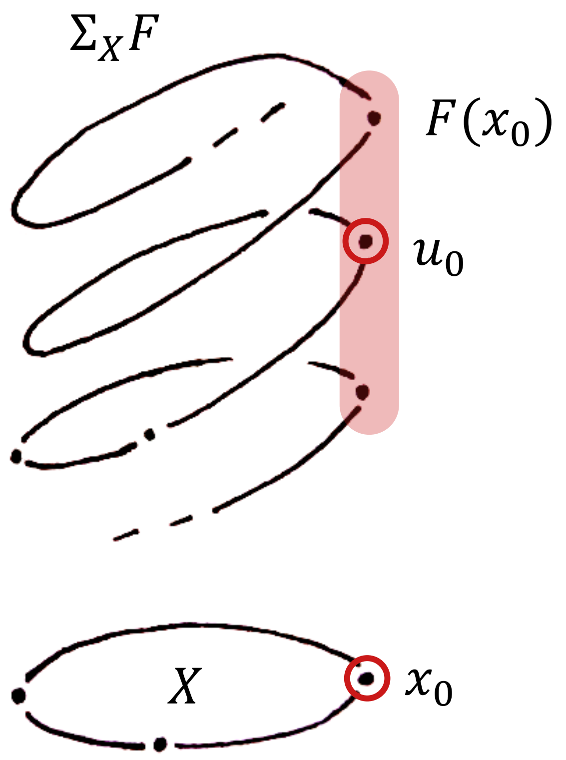

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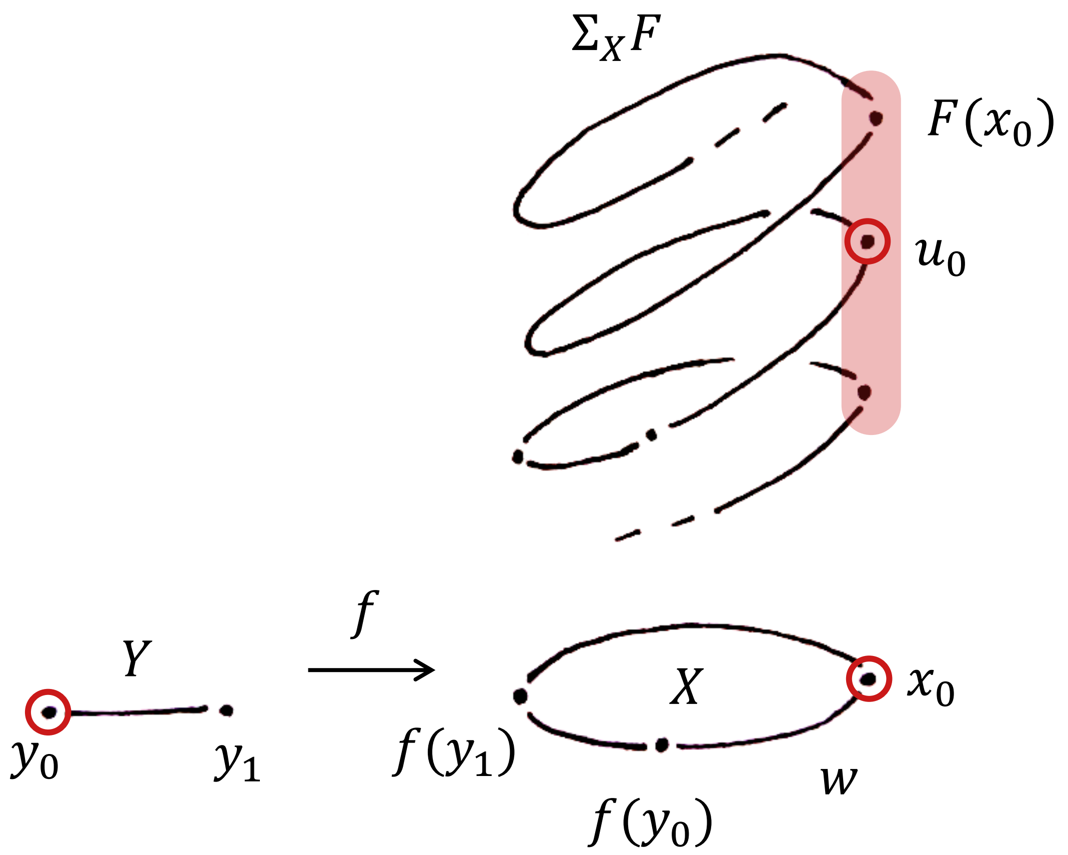

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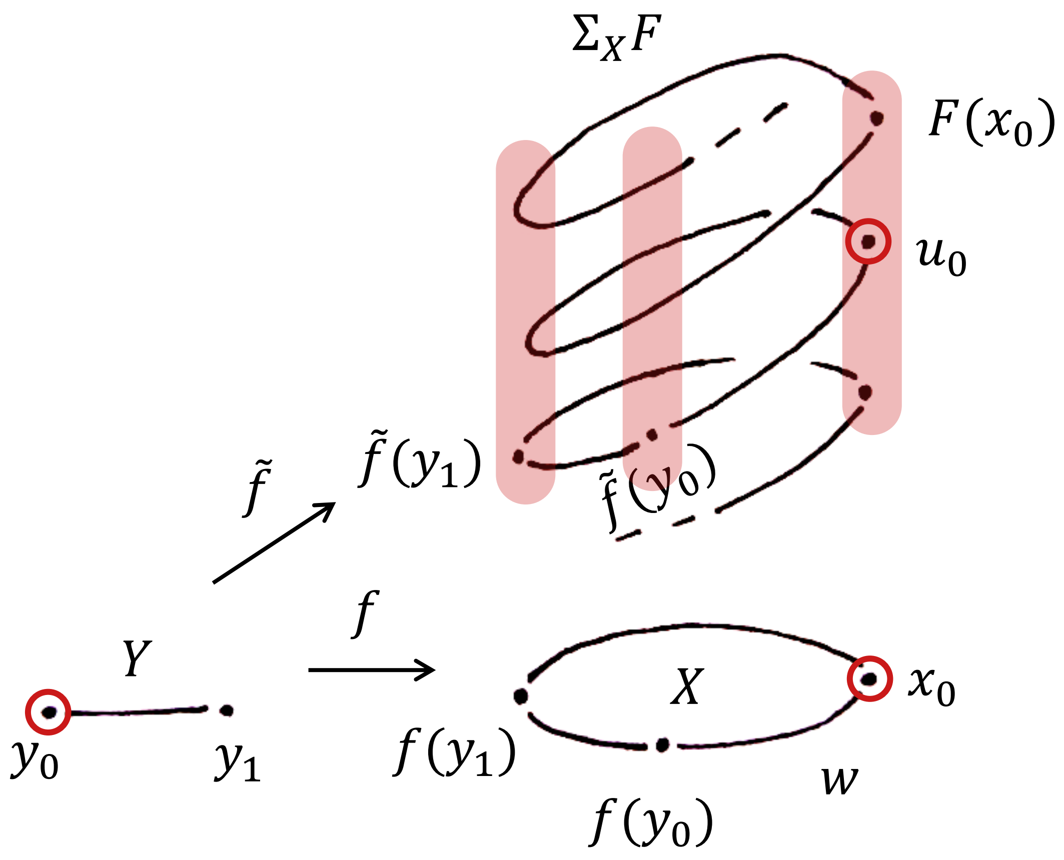

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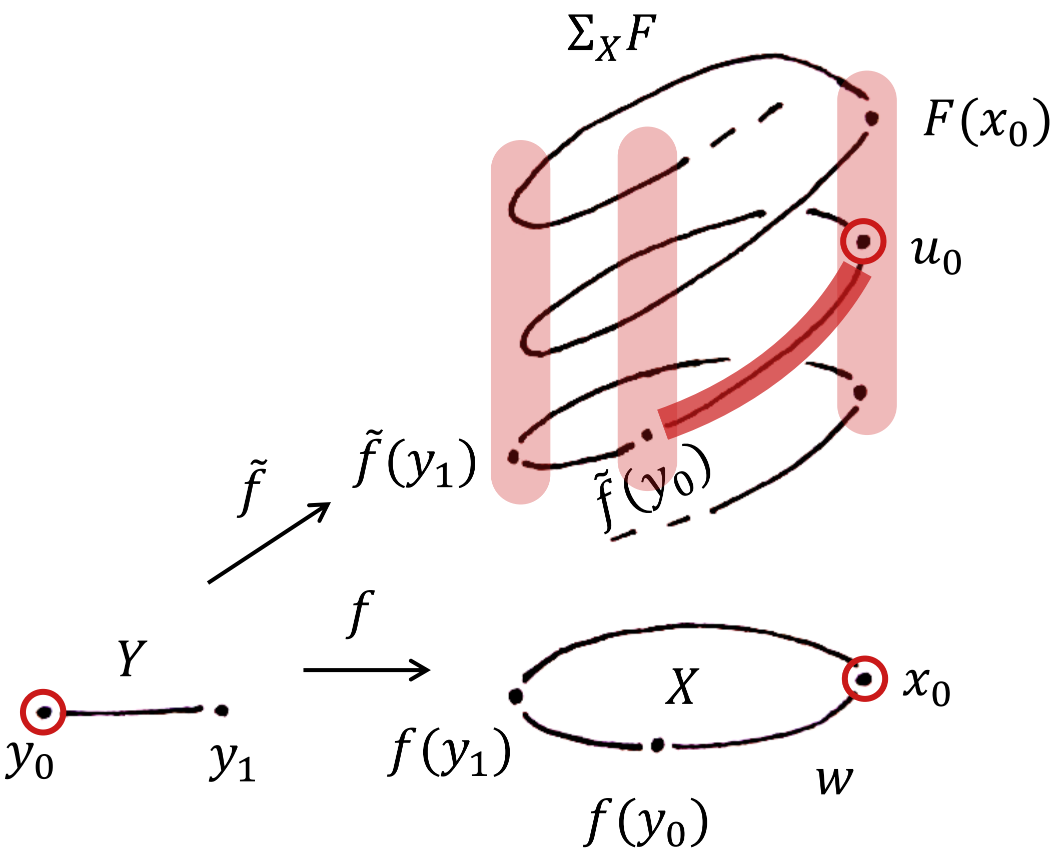

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Lemma

*Suppose given a covering space $F : X \to$ **Set** with point $u : F(x_0)$ over a pointed type (X, x_0) and a pointed map $f : (Y, y_0) \to (X, x_0)$ with Y connected. Then a pointed lift $\tilde{f}$: $\prod_{y:Y} F(f(y))$ of f exists iff*

$$f_*(\pi_1(Y,y_0)) \subset (\mathrm{pr}_1)_*\left(\pi_1\left(\Sigma_X F,(x_0;u_0)\right)\right)$$

About this criterion…

$$f_*(\pi_1(Y, y_0)) \subset (\mathrm{pr}_1)_* \left(\pi_1\left(\Sigma_X F, (x_0; u_0)\right)\right)$$

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- Conceals multiple **truncations** $\left\|\sum_{q:\|y_0=y_0\|_0} f_*(q) = p\right\|$
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for all loops of the form $f_*(p) : x_0 =_X x_0$ *with* $p : y_0 =_Y y_0$

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Lemma (version 2)

Suppose given a covering space $F : X \to \mathbf{Set}$ *with point* $u : F(x_0)$ *over a pointed type* (X, x_0) *and a pointed map* $f : (Y, y_0) \to (X, x_0)$ *with* Y *connected. Then a pointed lift* $\tilde{f}\colon \prod_{y:Y} F(f(y))$ *of* f *exists iff for all loops* $p : y_0 =_Y y_0$ *there exists a loop from* u_0 *to* u_0 *in* F *lying over* $f_*(p)$*, i.e.*

$$\mathrm{transport}^F(f_*(p), u_0) =_{F(x_0)} u_0$$

Proof closely reflects the **classical proof**

Canonical Change of Basepoint

Classical setting

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Path from a to b induces a **change-of-basepoint isomorphism**

$$\pi_n(X, a) \cong \pi_n(X, b)$$

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HoTT

Path $p : a =_X b$ also induces a change-of-basepoint isomorphism

$$\pi_n(X,a) \cong \pi_n(X,b)$$

via transport

- **Issue** X **connected**, then only $\|a =_X b\|$, so only

$$\|\pi_n(X,a) \cong \pi_n(X,b)\|$$

- **Wanted** an explicit isomorphism $\pi_n(X,a) \cong \pi_n(X,b)$

Via **extension by weak constancy** [Hou (Favonia) and Harper (2016)]
gives **canonical** isomorphism, if

For all paths $p, q : a =_X b$

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Theorem

Let X be a type with designated point $a : X$.

1. *If X is **simply-connected**, then the action of $\pi_1(X,a)$ on $\pi_n(X,a)$ is trivial for all $n \geq 1$*
2. *The fundamental group $\pi_1(X,a)$ is **abelian** if and only if the action on itself is trivial*
3. *If **merely** for all loops $p, q : \Omega(X,a)$, $p \cdot q = q \cdot p$ then the action of $\pi_1(X,a)$ on $\pi_n(X,a)$ is trivial for all $n \geq 1$*

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Lessons learned for formalization

- Work with fibers directly instead of the total space
- Remove truncations **… not always possible**

References

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