Lax-Idempotent 2-Monads, Degrees of Relatedness, and Multilevel Type Theory

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TYPES '23
Valencia, Spain
June 12, 2023

Degrees of Relatedness (RelDTT)

Nuyts and Devriese (2018) @ LICS

Degrees of Relatedness: Overview

- Parametricity is about **relations**,
- Equip types with **multiple**, **proof-relevant relations** $s \frown_i t$ indexed by **degree** i:
  - Just **one** for **small types** (Bool, $\mathbb{N} \to \mathbb{N}$, ...),
  - **More** for **larger** types ($U_0 \to U_0$, Grp, ...).
  - Proofs called ***i*-edges**.
- Describe **function behaviour** by saying how functions **influence degree** of relatedness,
- This explains
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Degrees of Relatedness: Four Laws

- **Reflexivity:** $(a : A) \frown_i^A (a : A)$
  (Semantically, prop. eq. = def. eq.)
- **Degradation:** $((a : A) \frown_i^R (b : B)) \to ((a : A) \frown_{i+1}^R (b : B))$
- **Dependency:** $(a : A) \frown_i^R (b : B)$ **presumes** $R : A \frown_{i+1}^{\mathrm{U}} B$
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Understanding degrees

$(a : A) \frown_0^A (b : A)$	Equality.
$(a : A) \frown_0^R (b : B)$	Heterogeneous equality along …
$R : (A : \mathsf{U}_0) \frown_1^{\mathsf{U}_0} (B : \mathsf{U}_0)$	Any relation R.
$P : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$	Any logical/algebraic relation P.
$Q : (G : \mathsf{Grp}) \frown_1^V (M : \mathsf{Monoid})$	Any logical/algebraic relation Q along …
$V : (\mathsf{Grp} : \mathsf{U}_1) \frown_2^{\mathsf{U}_1} (\mathsf{Monoid} : \mathsf{U}_1)$	V could specify that Q must respect e **and** $*$ (but it could ask Q to be anything).

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Understanding modalities (2)

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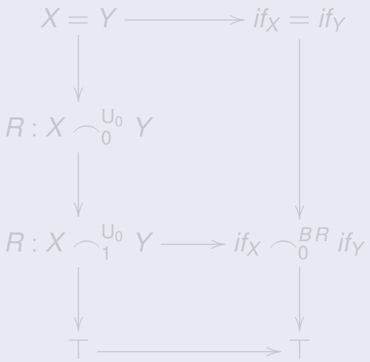
$$\mathit{if} : (\mathbf{par} \shortmid X : \mathsf{U}_0) \to B\, X$$



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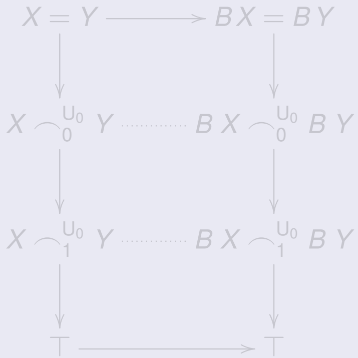

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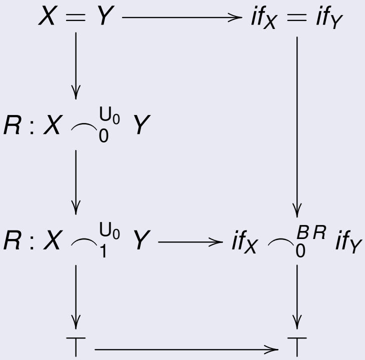

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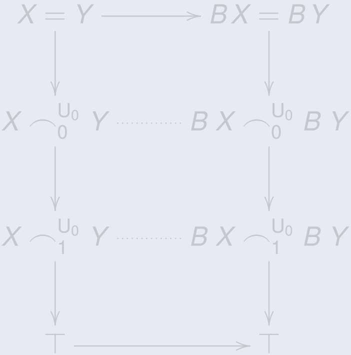

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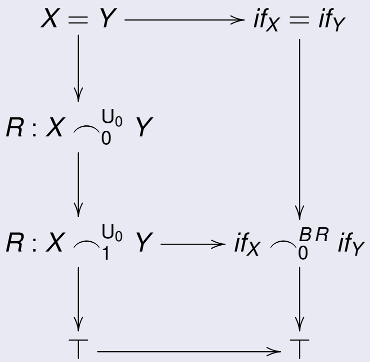

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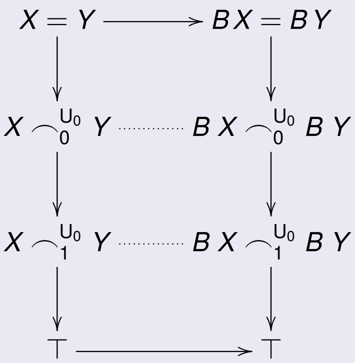

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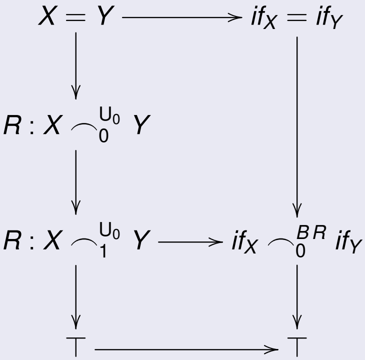

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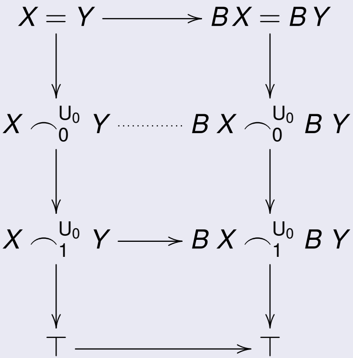

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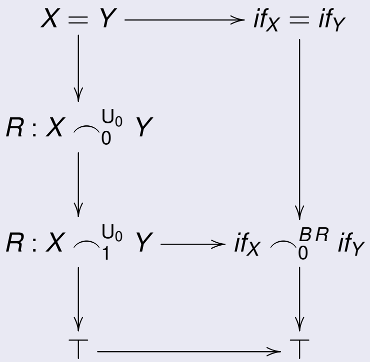

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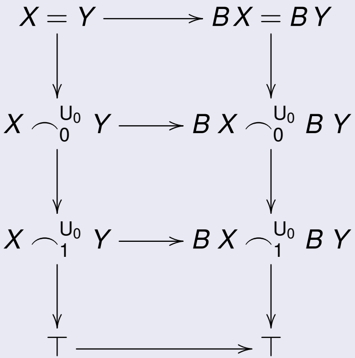

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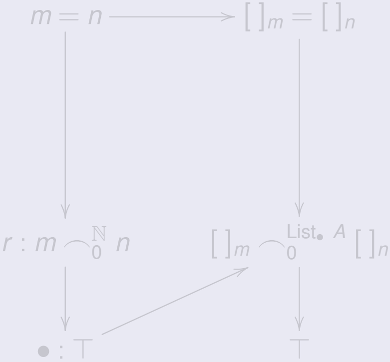

shi : values → types

$$\lambda n.\mathrm{List}_n\, A : (\mathbf{shi} \mid n : \mathbb{N}) \to \mathrm{U}_0$$

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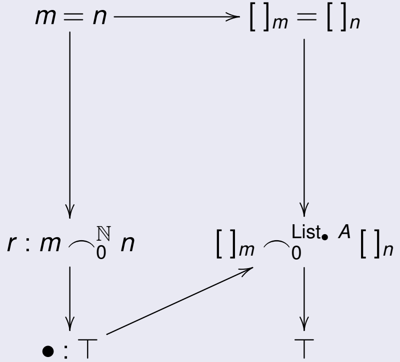

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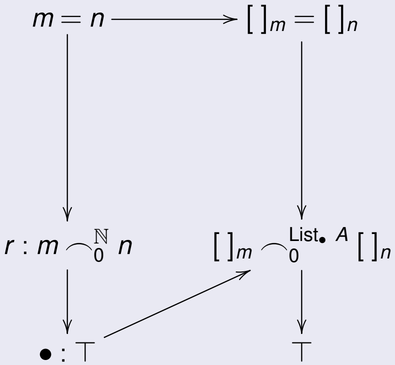

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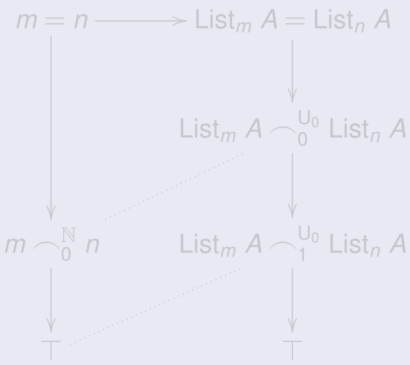

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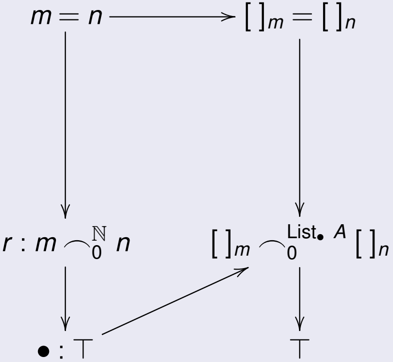

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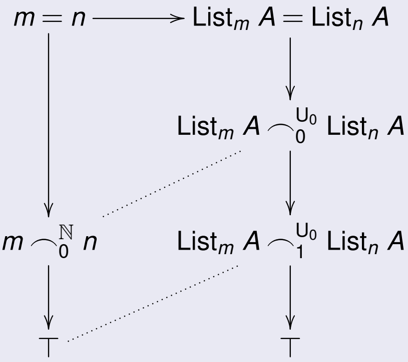

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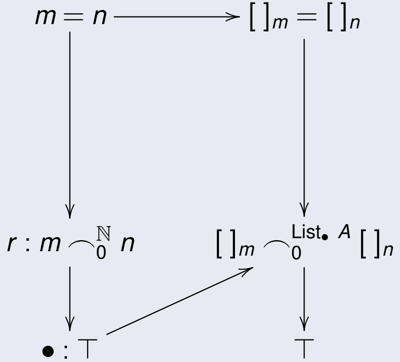

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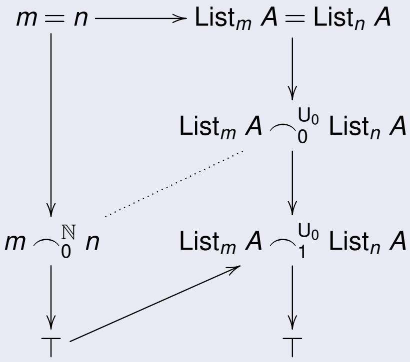

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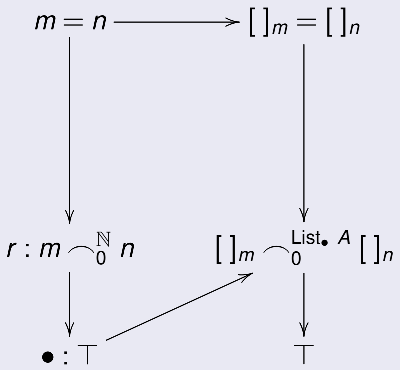

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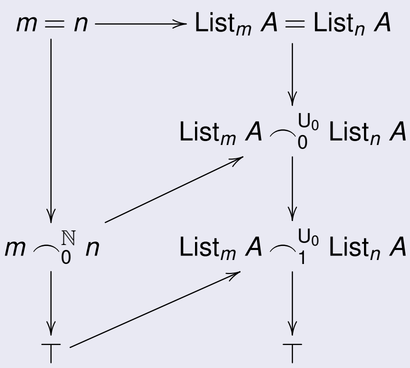

The mode theory DoR

RelDTT can be built on an instance of MTT (Multimode Type Theory) with mode theory DoR:

- Modes are **depths** $p \in \mathbb{Z}_{\geq -1}$
- Modalities $\mu : p \to q$ are
  functions $\{0 \leq \ldots \leq q\} \to \{(=) \leq 0 \leq \ldots \leq p \leq \top\} : i \mapsto i \cdot \mu$
  where $f : (\mu \mid x : A) \to B(x)$ sends

$$(r : x \frown^{A}_{i \cdot \mu} y) \quad \to \quad f(x) \frown^{B(r)}_{i} f(y).$$

  Modal types:

$$\mathrm{mod}_\mu\, x \frown^{\langle \mu | A \rangle}_{i} \mathrm{mod}_\mu\, y \quad = \quad x \frown^{A}_{i \cdot \mu} y$$

- 2-cells are degree-wise inequalities.

Depth p is modelled in cubical sets with $p+1$ different dimension flavours.

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- Modalities $\mu : p \to q$ are
  functions $\{0 \leq \ldots \leq q\} \to \{(=) \leq 0 \leq \ldots \leq p \leq \top\} : i \mapsto i \cdot \mu$
  where $f : (\mu \mid x : A) \to B(x)$ sends

$$(r : x \frown^{A}_{i \cdot \mu} y) \quad \to \quad f(x) \frown^{B(r)}_{i} f(y).$$

Modal types:

$$\text{mod}_{\mu}\, x \frown^{\langle \mu | A \rangle}_{i} \text{mod}_{\mu}\, y \quad = \quad x \frown^{A}_{i \cdot \mu} y$$

- 2-cells are degree-wise inequalities.

Depth p is modelled in cubical sets with $p + 1$ different dimension flavours.

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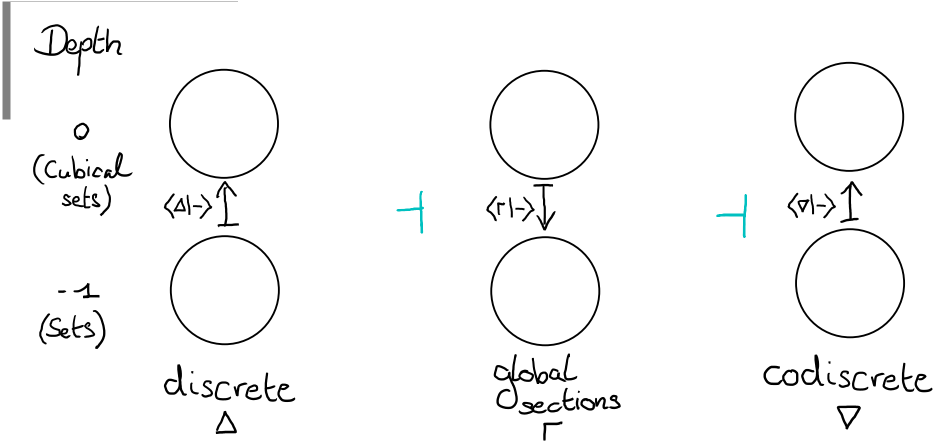
Depth

0
(Cubical sets)

-1
(Sets)

Cohesion in RelDTT



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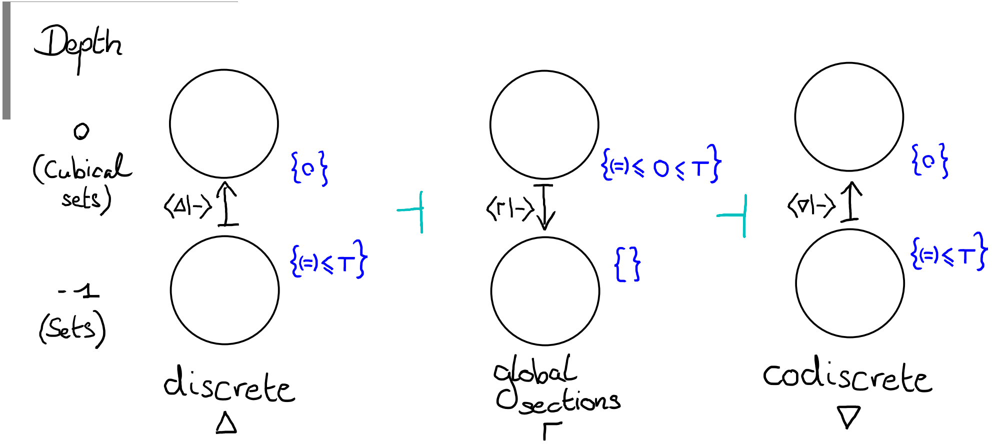

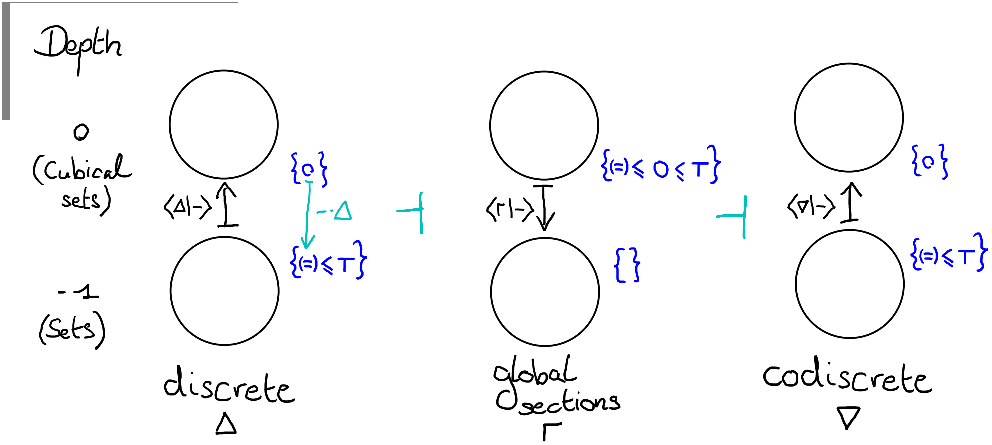
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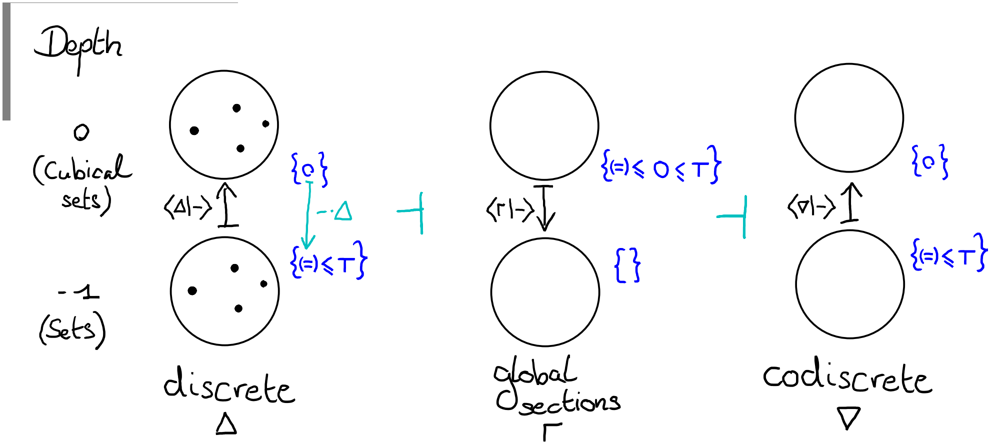

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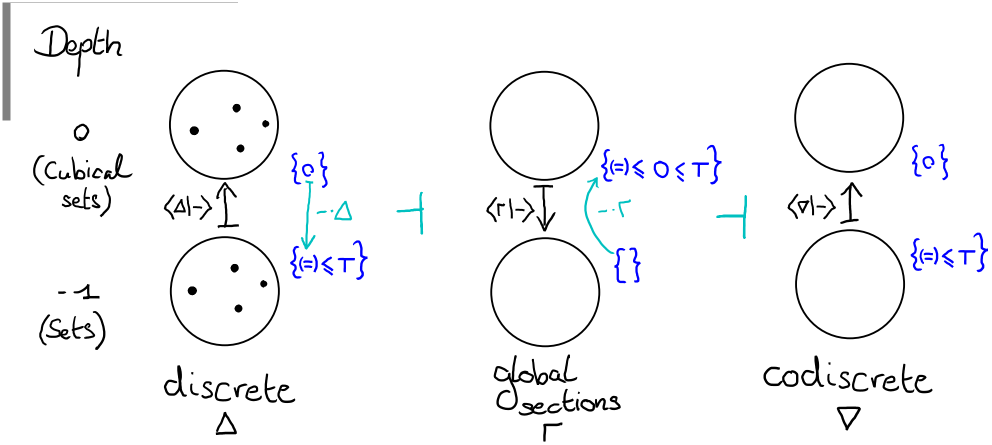

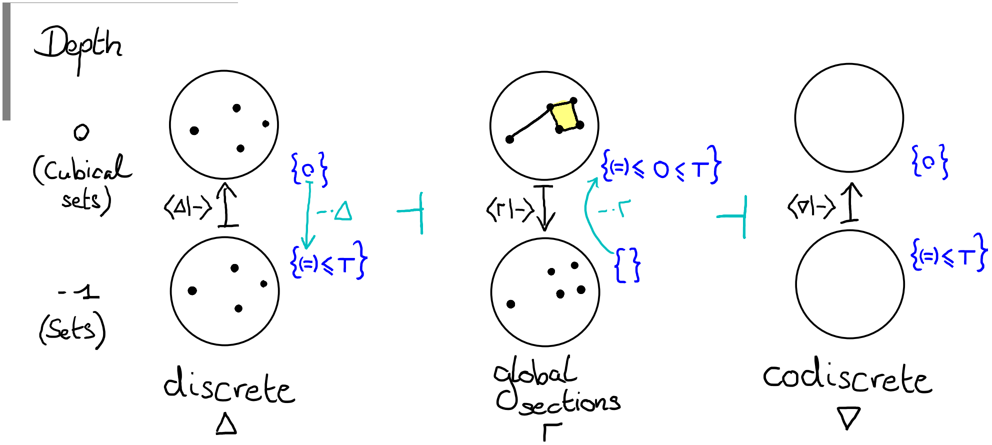

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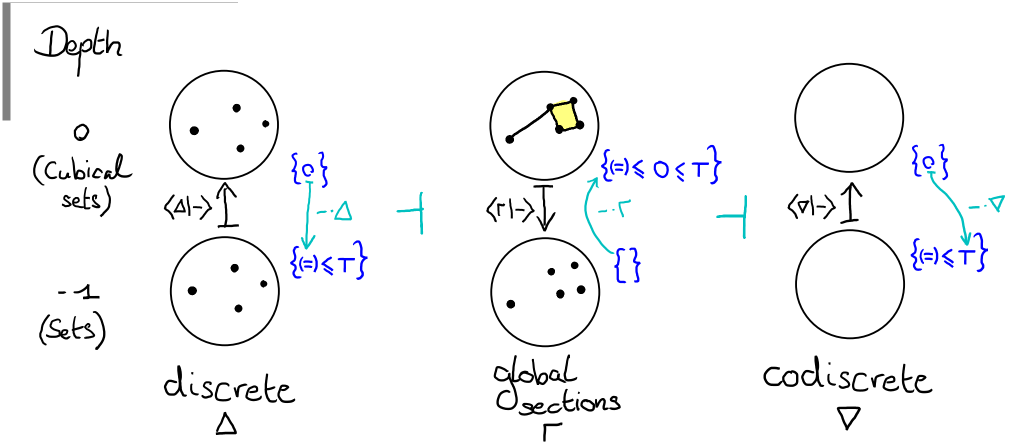

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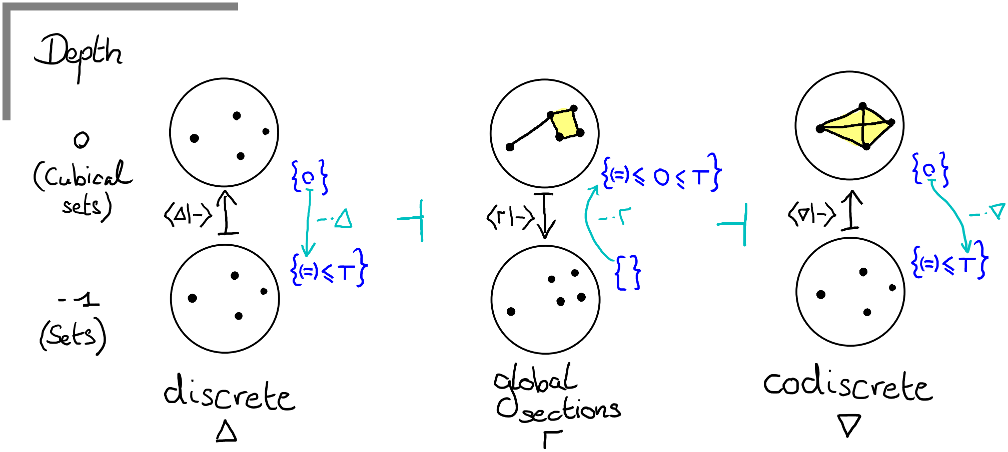

Two-Level Type Theory (2LTT)

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Altenkirch, Capriotti & Kraus (2016)
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More specifically, **its general presheaf model**

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CwF $(\mathscr{C}, \mathrm{Ty}^i, \mathrm{Tm}^i)$

$\mathrm{Ty}^i \in \mathrm{Psh}(\mathscr{C})$

$\mathrm{Tm}^i \in \mathrm{Psh}\left(\int_{\mathscr{C}} \mathrm{Ty}^i\right)$

$\Gamma\ \mathsf{ctx}^i$

$\Gamma \vdash^i T\ \mathsf{type}$

$\Gamma \vdash^i t : T$

E.g inner system = HoTT

Outer Level

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Talk about strict equality

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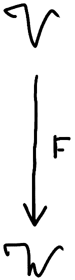
Lifting Functors

Multilevel Type Theory



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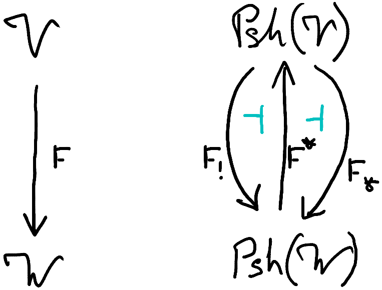
Lifting Functors



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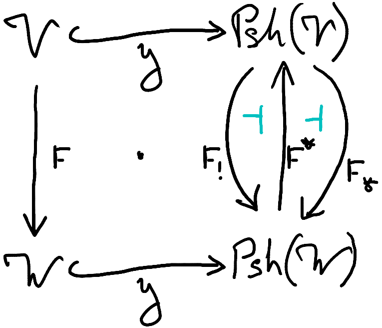

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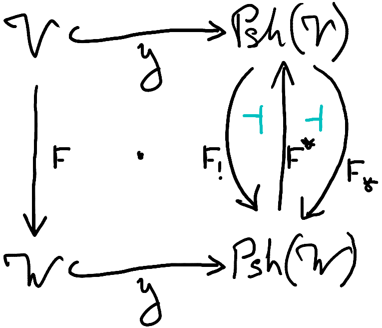

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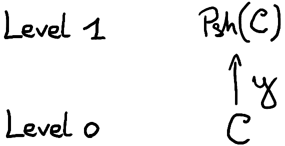

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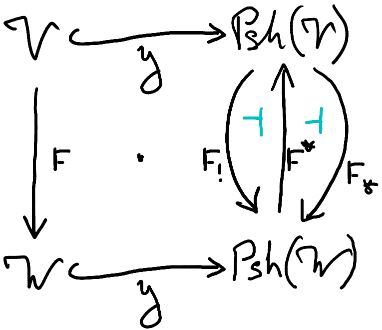

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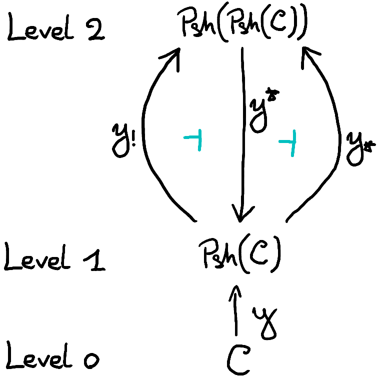

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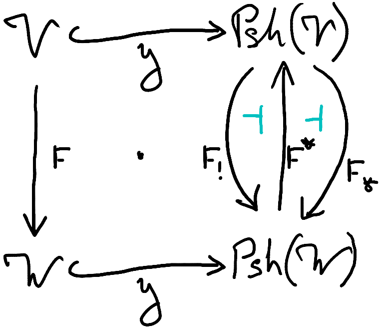

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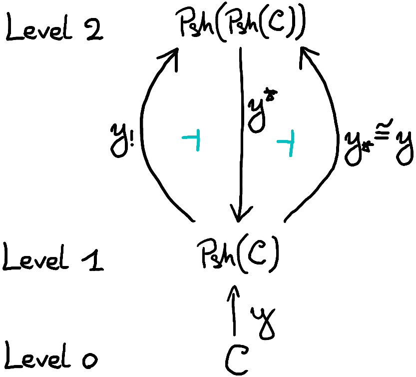

Multilevel TT: Case Study

- Level 0: $\star$ (point category is a CwF)
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What is $\mathrm{Psh}(\mathsf{Set})$?

Presheaves over
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with ∞-dimensional simplices

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  - $\mathbf{y}^* : \mathrm{Psh}(\mathsf{Set}) \to \mathsf{Set}$ extracts points.

What is $\mathrm{Psh}(\mathsf{Set})$?

Presheaves over

~~finite~~

~~non-empty~~

~~linear orders~~ sets

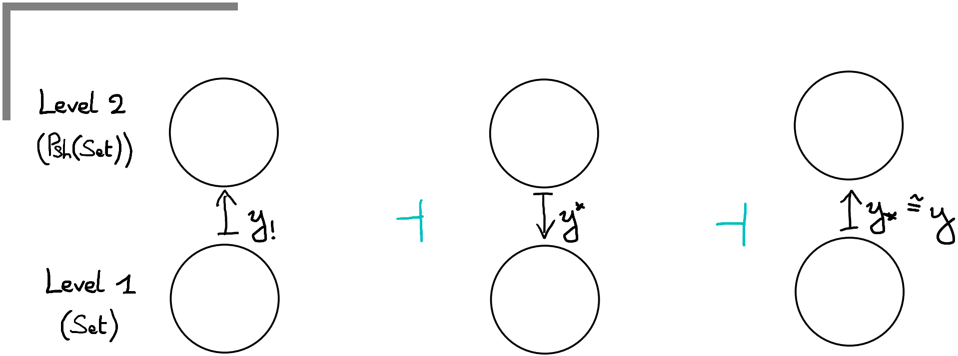
are

symmetric

~~semi~~ augmented simplicial sets

with ∞-dimensional simplices

Cohesion in Multilevel TT



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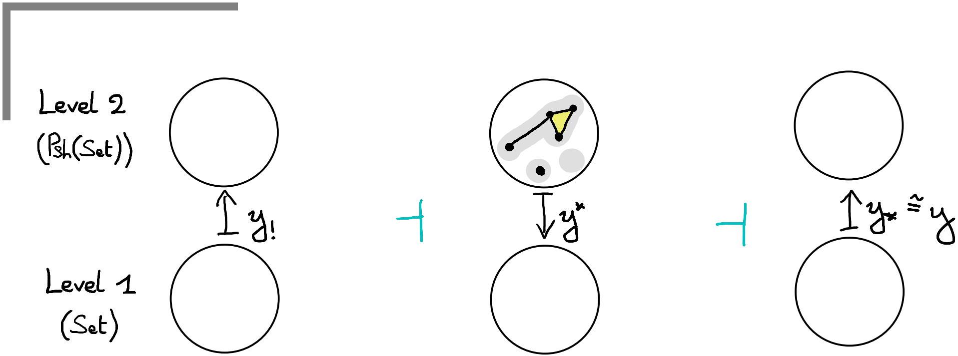

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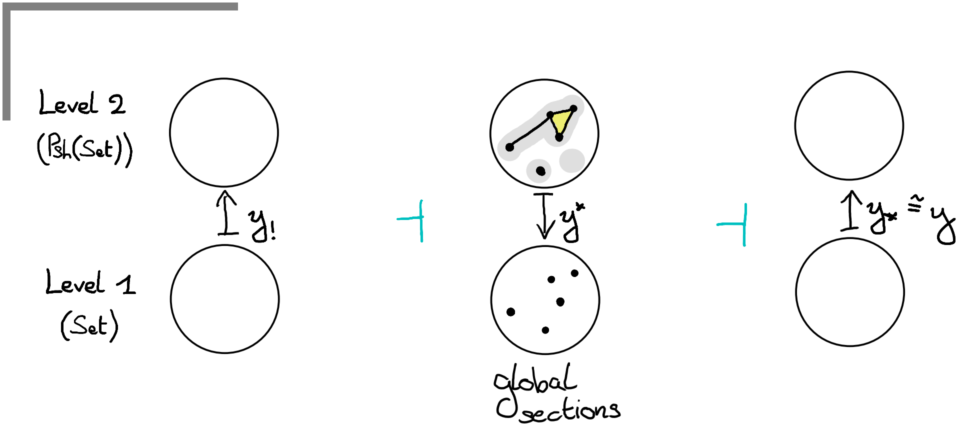

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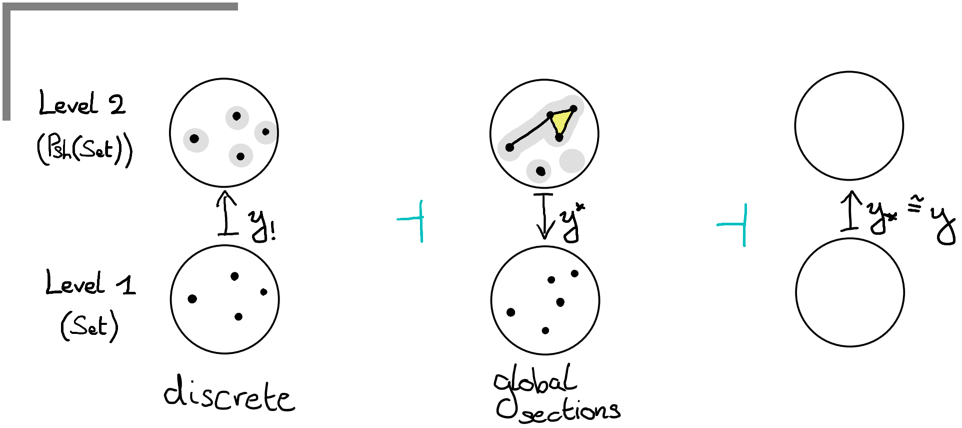

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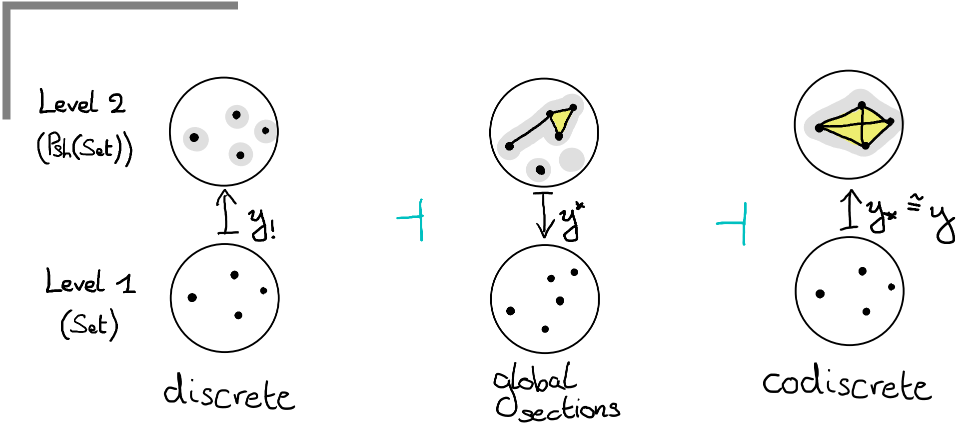

Degrees of Relatedness $\overset{?}{\sim}$ Multilevel TT

Goal: Formalize this correspondence, using **interface** of

Lax-idempotent 2-monads

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Goal: Formalize this correspondence, using **interface** of

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Why Psh : Cat → Cat is a Monad

- In Psh(Set), Yoneda-embeddings (representables) are **codiscrete**.
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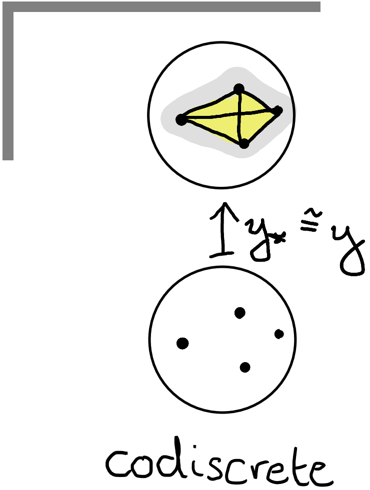

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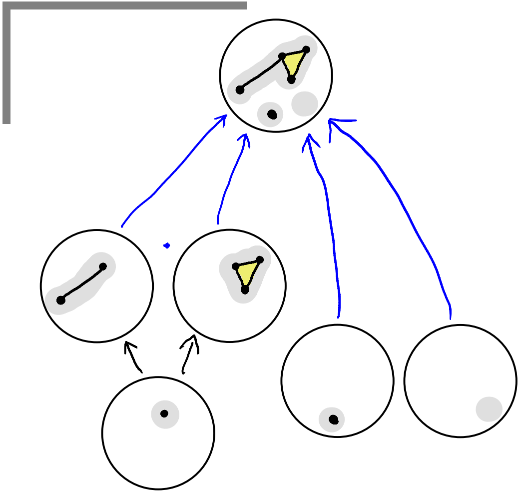

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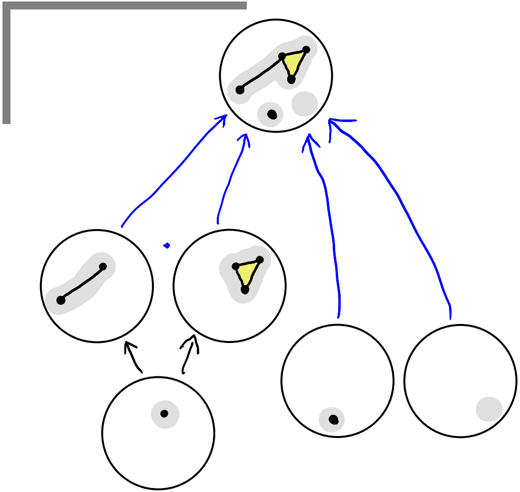

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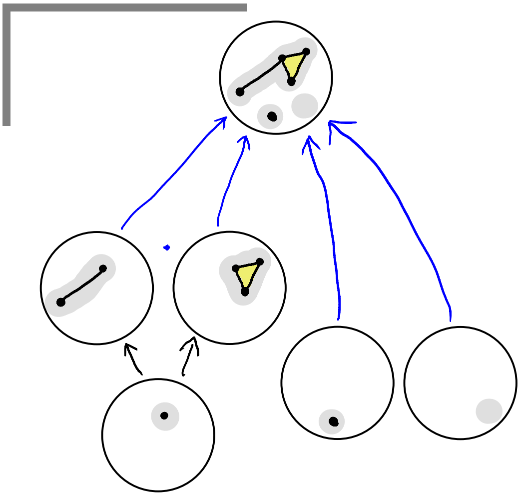

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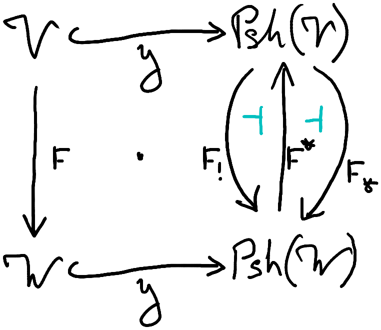

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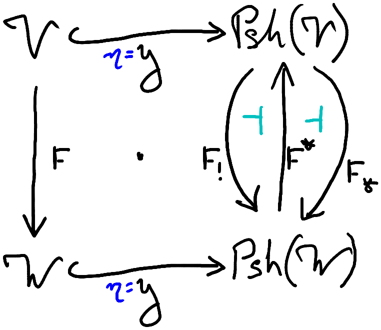

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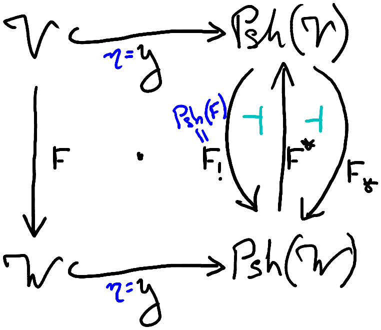

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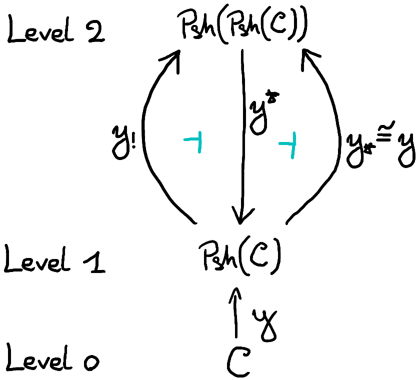

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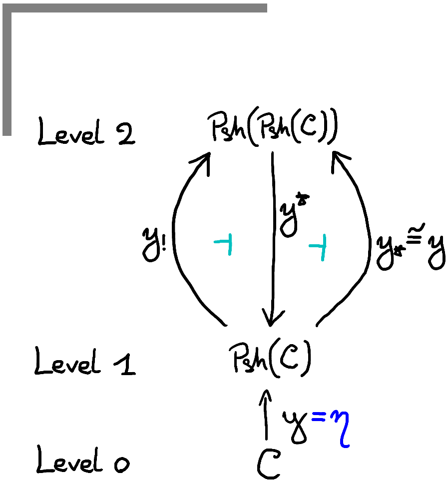

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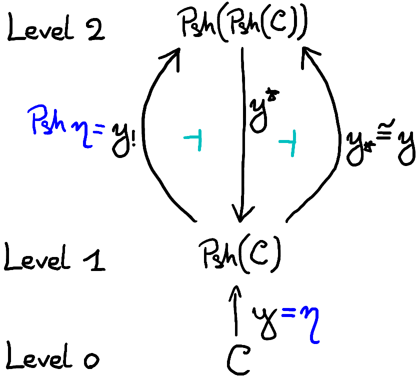

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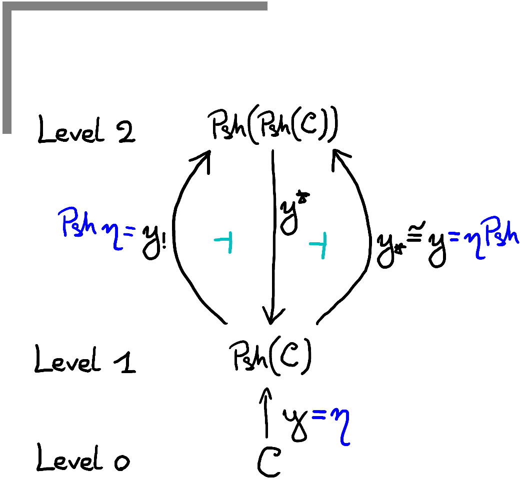

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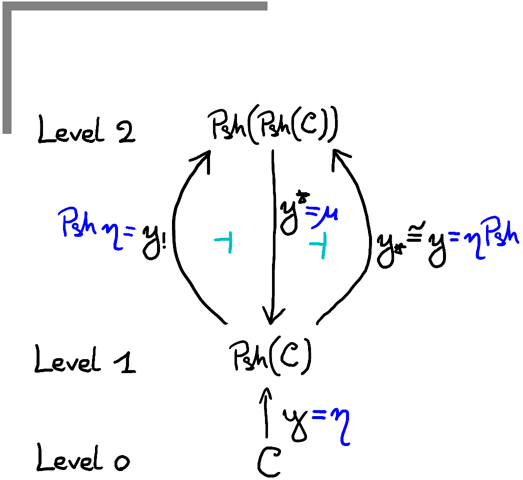

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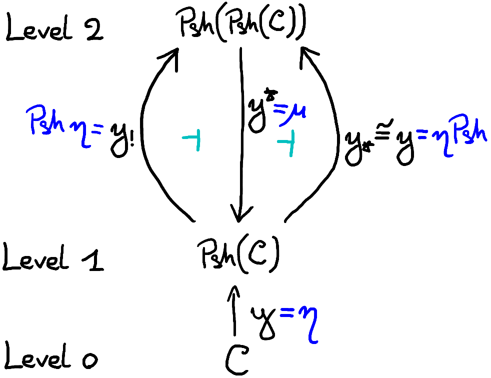

Definition

Define 2-category DoR:

- Objects are $p \in \mathbb{Z}_{\geq -1}$
- Morphisms $\mu : p \to q$ are functions $\{0 \leq \ldots \leq q\} \to \{(=) \leq 0 \leq \ldots \leq p \leq \top\} : i \mapsto i \cdot \mu$
- 2-cells are degree-wise inequalities.
- Freely add $\bot =: -2$.

Definition

Let LIM be the 2-category freely generated by:

- $\mathbf{C} \in \mathrm{Obj}(\mathrm{LIM})$,
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Clearly, we get $[\![-]\!]_{\mathscr{C}} : \mathrm{LIM} \to \mathrm{Cat}$ for any category $\mathscr{C}$:

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Main theorem (formal proof WIP)

$$\begin{array}{ccccc} \mathrm{DoR} & \cong & \mathrm{LIM} & \to & \mathrm{Cat} \\ p & \mapsto & \mathbf{M}^{p+2}(\mathbf{C}) & \mapsto & \mathrm{Psh}^{p+2}(\mathscr{C}) \end{array}$$

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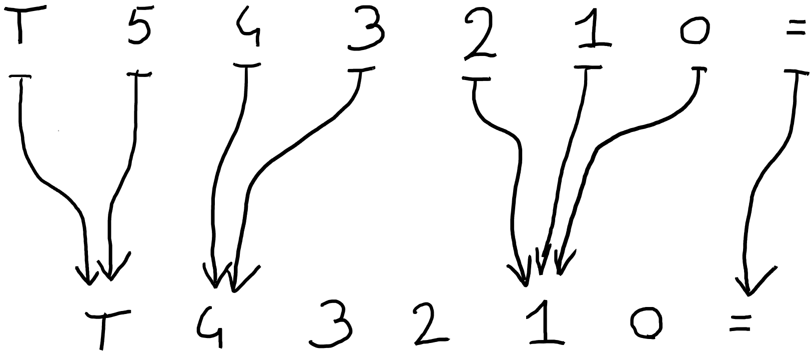
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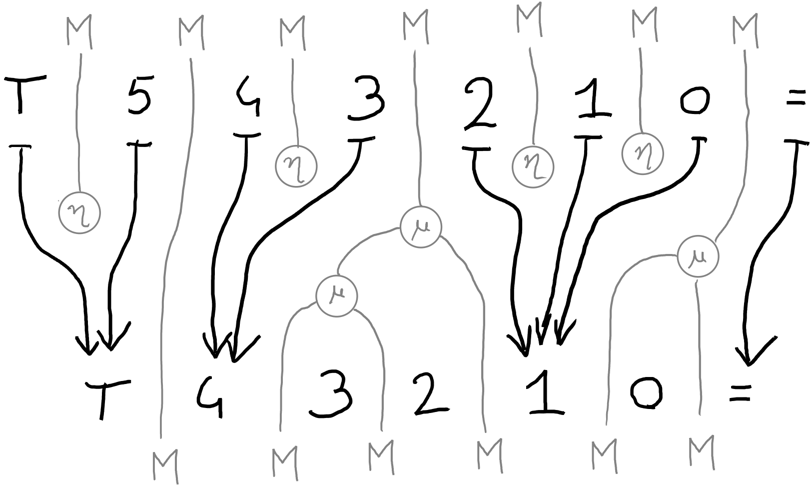
Sketch of Proof



Conclusion

Implications of main theorem:

- **Degrees of Relatedness*** can serve as an **internal language** for **multilevel type theory**,
- A model for **parametricity, irrelevance, . . .** found **in the wild**.

To do:

- Formalize proof.
- Study **discreteness** and **internal parametricity** in this setting.
- Directify :-)

*Or a reasonable adaptation.

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Thanks!

Questions?