Read the mode and stay positive

Towards internal positivity annotations

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data ℕ : Set where
  zero : ℕ
  succ : ℕ → ℕ
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- Cons: induction scheme lives outside of the type theory, can't abstract over it.

W-types

First internal approximation: W-types[1] aka. containers[2].

[1] Martin-Löf, “Intuitionistic type theory”.
[2] Abbott, Altenkirch, and Ghani, “Categories of Containers”.

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First internal approximation: W-types[1] aka. containers[2].

Define the following primitive

```
data W (S : Set) (P : S → Set) : Set where
  sup : (cons : S) → (P cons → W S P) → W S P
```

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S is the type of *shapes* while P is the type of *positions*, ie. arities of the recursive references of the constructors.

[1]Martin-Löf, "Intuitionistic type theory".

[2]Abbott, Altenkirch, and Ghani, "Categories of Containers".

```
ℕShapes : Set
ℕShapes = Bool

ℕPositions : ℕShapes → Set
ℕPositions (inl tt) = ⊥ -- zero constructor
ℕPositions (inr tt) = ⊤ -- succ constructor

ℕ = W ℕShapes ℕPositions
```

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```
List : Set → Set
List A = W ListShapes ListPositions
  where ListShapes : Set
        ListShapes = ⊤ ⊎ A

        ListPositions : ListShapes → Set
        ListPositions (inl _) = ⊥
        ListPositions (inr _) = ⊤
```

We have one shape per element of A!

```
def-naturals : (n : ℕ)
             → (n ≡ zero)
               ⊎ (Σ[ m ∈ ℕ ] (n ≡ (succ m)))
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def-naturals (sup (inl tt) f) = {!!}
-- need to show f ≡ (λ ())
def-naturals (sup (inr tt) g) = {!!}
-- need to show g ≡ (λ tt → g tt)
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`funExt` needed!

Fixed points

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```
ℕ-type-former : Set → Set
ℕ-type-former X = ⊤ ⊎ X
```

How do we take least fixed points of type formers?

```
μ : (Set → Set) → Set
μ F = {!!}
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Not all type formers have fixed points!

```
oops-type-former : Set → Set
oops-type-former X = X → ⊥
```

Proof assistants already use a syntactic criterion: strict positivity.

```
data I : Set where
  ok : (⊥ → I) → I → I
  not-ok : {! ((I → ⊥) → ⊥) → I !}
```

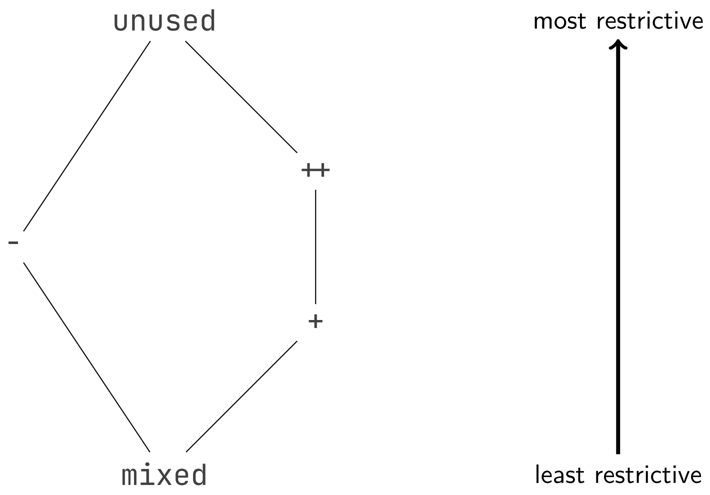
However, we have no way of saying inside the type system that a type former is strictly positive, so we still can't type μ.

Let's just extend the type system then!

```
id : @++ Set → Set
id X = X

not-positive : @++ Set → Set
not-positive X = {! X → ⊥ !}
```

New modal type theory, inspired by Abel, "Polarized Subtyping for Sized Types" and following the framework of Gratzer et al., "Multimodal Dependent Type Theory".



unused
most restrictive
++
-
+
mixed
least restrictive

```
spositive : @++ Set → Set
spositive A = ⊥ → A

negative : @- Set → Set
negative A = A → ⊥
```

Modalities compose!

```
f : @+ Set → Set
f A = negative (spositive (negative A))
```

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f A = negative (spositive (negative A))
```

Very importantly

```
pi : (@- A : Set) → (@++ B : Set) → Set
pi A B = A → B
```

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```
data μ (@++ F : @++ Set → Set) : Set where
  fix : F (μ F) → μ F
```

Back to our running examples:

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ℕ : Set
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List A = μ (λ X → ⊤ ⊎ A × X)
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Back to our running examples:

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ℕ : Set
ℕ = μ (λ X → ⊤ ⊎ X)
```

```
List : @++ Set → Set
List A = μ (λ X → ⊤ ⊎ A × X)
```

```
W : (A : Set) (B : A → Set)
  → @++ Set → Set
W A B X =
  Σ[ a ∈ A ] (B a → X)
```

```
pattern zero = fix (inl tt)
pattern succ n = fix (inr n)

pattern nil = fix (inl tt)
pattern cons a b = fix (inr (a , b))

length : {A : Set}
       → List A
       → ℕ
length nil = zero
length (cons _ b) = succ (length b)
```

Further work

Subtyping

```
id : (@++ Set → Set) → (Set → Set)
id F = {!F!}
```

Need to eta-expand!

Reduces composability.

Generic fmap

```
fmap : (F : @+ Set → Set) {A B : Set}
     → (f : A → B)
     → (F A → F B)
fmap f fa = {!!}
```

Sound familiar?

Generic fmap

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fmap : (F : @+ Set → Set) {A B : Set}
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Sound familiar?

```
equivmap : (F : Type → Type) {A B : Type}
         → (eq : A ≃ B)
         → F A ≃ F B
equivmap F eq = pathToEquiv (cong F (ua eq))
```

```
{-# TERMINATING #-}
μelim : (F : @++ Set → Set) {A : Set}
      → (alg : F A → A)
      → (μ F → A)
μelim F alg (fix x) = alg (fmap F (μelim F alg) x)
```

→ Internalize some generic programming.

Directed Type Theory

```
data Hom[_,_] {A : Set ℓ}
      : @- A → @+ A → Set ℓ
      where
  id : ∀ {@unused x} → Hom[ x , x ]
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elim : {A : Set ℓ}
         (F : @- A → @+ A → Set ℓ')
      → ((@unused x : A) → F x x)
      ------------------------------
      → (@unused x y : A)
      → (Hom[ x , y ] → F x y)
elim F F-id x .x id = F-id x
```

```
fmap : {A B : Set ℓ}
     → (F : @+ Set ℓ → Set ℓ')
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fmap : {A B : Set ℓ}
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HomToFun : {A B : Set ℓ}
         → Hom[ A , B ] → A → B
HomToFun {A = A} {B = B} =
  elim (λ X Y → X → Y) (λ X x → x) A B
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```
postulate ua : {A B : Set ℓ}
             → (A → B) → Hom[ A , B ]
```

Replacing positivity checks entirely

Remove syntactic positivity check and replace with type checking.

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Add μ as a primitive and make data declarations desugar into a use of it.

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Add μ as a primitive and make data declarations desugar into a use of it.

But we still don't know exactly how it interacts with funky inductive types like inductive-inductive or inductive-recursive ones!

Annotation inferrer

Recontextualize positivity checking as an annotation elaboration algorithm.

$\rightarrow$ Flexibility of type annotations + comfort of automation.

Semantics

- We're now in directed type theory, so we need a directed model like in North, "Towards a Directed Homotopy Type Theory".

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Fix-point mode: categorical analogue of dcpos, $< \varkappa$-locally presentable categories with $< \varkappa$-accessible functors between them.

Complication: The rank of accessibility of $-^A$ depends on the cardinality of A.

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Fix-point mode: categorical analogue of dcpos, $< \varkappa$-locally presentable categories with $< \varkappa$-accessible functors between them.

Complication: The rank of accessibility of $-^A$ depends on the cardinality of A.

No clear semantics for the so-called lock operator on contexts in MTT for our modalities.

Takeaway

A prototype Agda implementation at `https://github.com/agda/agda/pull/6385`. Some work already merged!

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A prototype Agda implementation at `https://github.com/agda/agda/pull/6385`. Some work already merged!

- Internalize induction schemes synthetically;
- Use type-checking instead of syntactical checks for inductive types;
- Real-world use of directed type theory for programmers!

Abbott, Michael, Thorsten Altenkirch, and Neil Ghani. "Categories of Containers". In: *Foundations of Software Science and Computation Structures*. Ed. by Andrew D. Gordon. Berlin, Heidelberg: Springer Berlin Heidelberg, 2003, pp. 23–38. ISBN: 978-3-540-36576-1.

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Thank you!

Any questions? Suggestions?