Higher coherence equations of semi-simplicial types as n-cubes of proofs

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TYPES 2023 — Valencia, Spain

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We still want to have an explicit construction of semi-simplicial types eventually…

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- …And nothing more!

Straightforward to do in type theory if we have UIP (or restrict ourselves to sets)

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Problem (Voevodsky and others, 2012)

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- Constructions are no longer set-truncated
- Higher coherence issues!

The semi-simplicial identity $d_i d_j = d_{j-1} d_i$ induces “higher proof terms” that interfere with each other and need to be identified

Illustration

Suppose we want to show that $d_i d_j d_k =_{X_n \to X_{n-3}} d_{k-2} d_{j-1} d_i$ in type theory

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- This is given by

$$\pi := \alpha_{j,k} \cdot \alpha_{i,k-1} \cdot \alpha_{i,j} \quad \text{and} \quad \pi' := \alpha_{i,j} \cdot \alpha_{i,k} \cdot \alpha_{j-1,k-1}$$

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- We now need the data of a term $\beta_{i,j,k} : \pi = \pi'$, and so on…

Possible approaches

- *n*-Truncate (e.g. **h-Set**s, **h-Grp**s, ...)

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$$\begin{aligned}
X_0 &: \textbf{Type} \\
X_1 &: X_0 \to X_0 \to \textbf{Type} \\
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A general solution is believed to be impossible in "plain" HoTT

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2022 MLTT + *Type Streams* (Kolomatskaia)

TLCA 2015 — Warsaw, Poland

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Strict composition

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The explicit form taken by the higher-order coherence equations is well described in the case of (strict) ω-categories, i.e. when associativity and the exchange law hold on the nose for path composition

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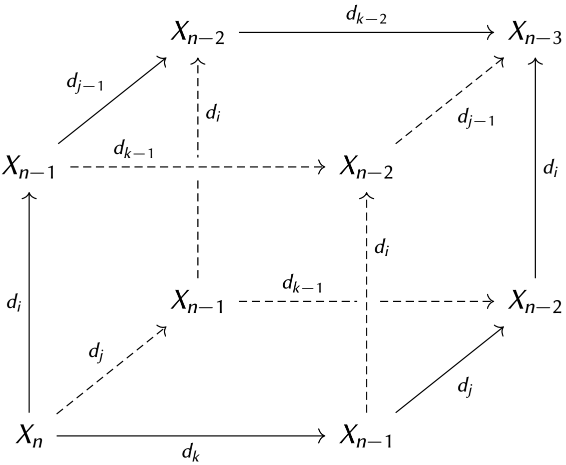
How to give a recursive formulation of all the compositions involved?

3-Cube of proofs



$$\beta_{i,j,k} : \overbrace{\alpha_{j,k} \cdot \alpha_{i,k-1} \cdot \alpha_{i,j}}^{\text{Backmost}} =_{\left(d_i d_j d_k =_{(X_n \to X_{n-3})} d_{k-2} d_{j-1} d_i\right)} \underbrace{\alpha_{i,j} \cdot \alpha_{i,k} \cdot \alpha_{j-1,k-1}}_{\text{Frontmost}}$$

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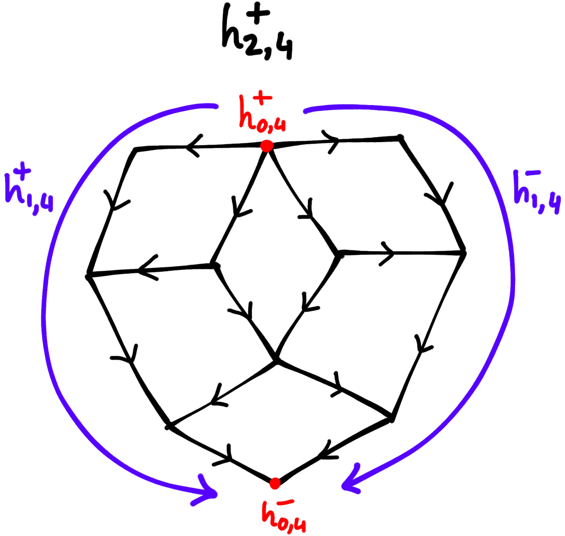
- We write $h^{\pm}_{k,n}$ for the k-hemispheres of the n-cube

Illustration



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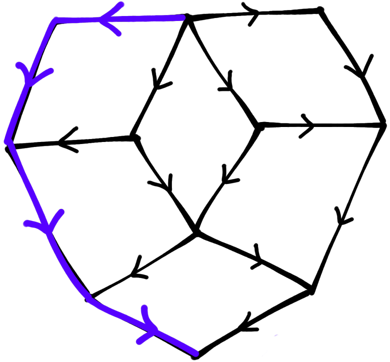

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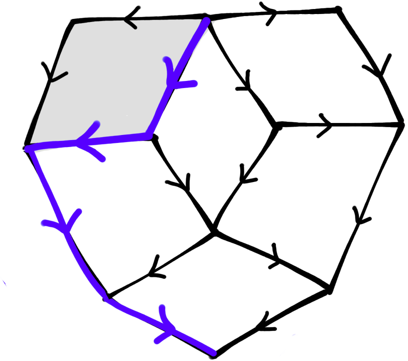

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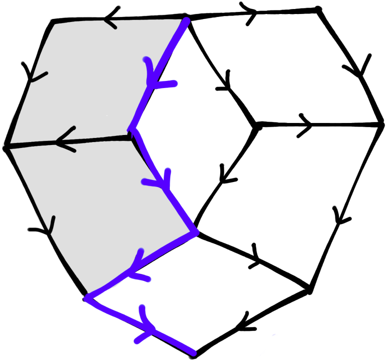

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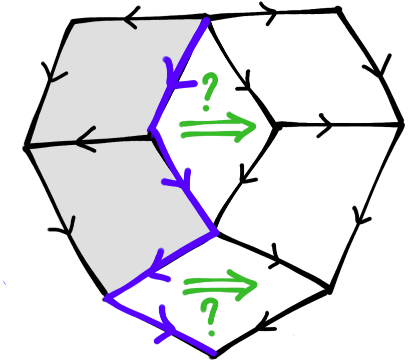

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D_k^n is the poset of *increasing* sequences $x_1 \ldots x_n$ of length n with values $0 \leq x_i \leq k$ equipped with the "pointwise" order

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Theorem

The k-hemispheres of the n-cube are described by D_{n-k}^k in the sense that there is a bijection sending any $h_{k,n}^{\pm}$ onto a linear extension of D_{n-k}^k

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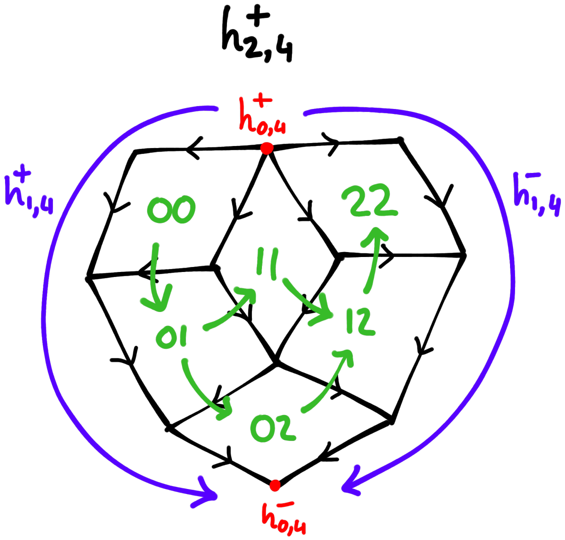

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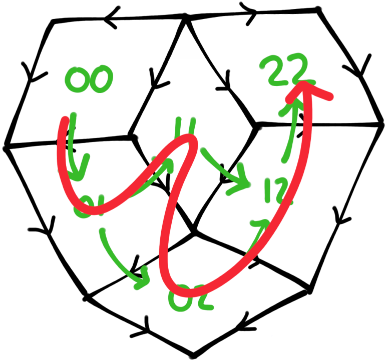

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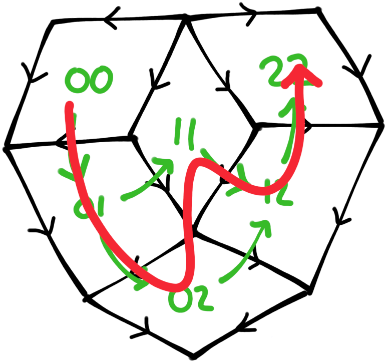

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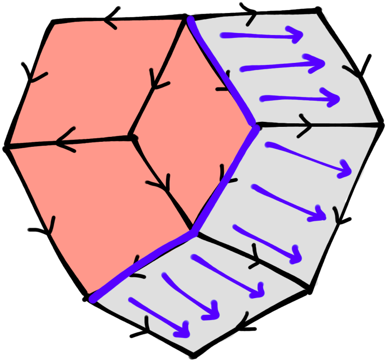
Observation

$h_{k,n}^{\pm}$ has a similar recursive formula

Extrusion of the 3-cube

$$h^+_{2,4} = \text{“}\, h^+_{2,3} + h^-_{1,3} \,\text{”}$$



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$$\square\ 3\ i\ j\ k \equiv \beta_{i,j,k}$$

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- Plug this into a "cumulatively defined" indexed construction (see appendix for an idea)
- Define semi-simplicial types in homotopy type theory

Thank you!

(Questions?)

Appendix

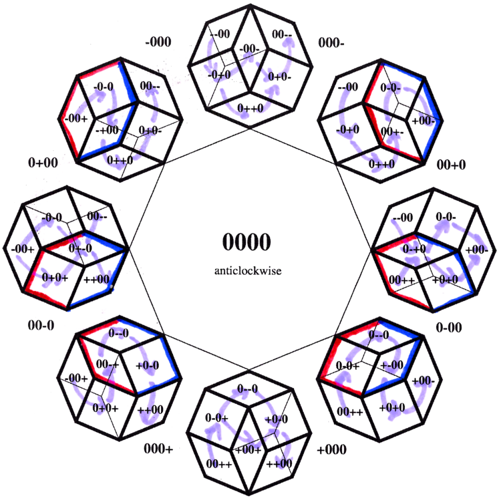
$$X_0 : A_0 \longrightarrow \textbf{Type}$$

$$X_1 : \forall\, a_1 : A_1,$$
$$\forall\, s_0 : \left[\prod_{a_0 : A_0} \prod_{f_0 : \mathrm{Hom}(a_0, a_1)} X_0\ a_0\right],$$
$$\textbf{Type}$$

$$X_2 : \forall\, a_2 : A_2,$$
$$\forall\, s_0 : \left[\prod_{a_0 : A_0} \prod_{f_0 : \mathrm{Hom}(a_0, a_2)} X_0\ a_0\right],$$
$$\forall\, s_1 : \left[\prod_{a_1 : A_1} \prod_{f_1 : \mathrm{Hom}(a_1, a_2)} X_1\ a_1\ (\lambda f_0\, .\, s_0(f_1 \circ f_0))\right],$$
$$\textbf{Type}$$

$$X_3 : \forall\, a_3 : A_3,$$
$$\forall\, s_0 : \left[\prod_{a_0 : A_0} \prod_{f_0 : \mathrm{Hom}(a_0, a_3)} X_0\ a_0\right],$$
$$\forall\, s_1 : \left[\prod_{a_1 : A_1} \prod_{f_1 : \mathrm{Hom}(a_1, a_3)} X_1\ a_1\ (\lambda f_0\, .\, s_0(f_1 \circ f_0))\right],$$
$$\forall\, s_2 : \left[\prod_{a_2 : A_2} \prod_{f_2 : \mathrm{Hom}(a_2, a_3)} X_2\ a_2\ (\lambda f_0\, .\, s_0(f_2 \circ f_0))\ (\lambda f_1\, .\, s_1(f_2 \circ f_1))\right],$$
$$\textbf{Type}$$

Appendix



“Octagon of octagons” — Extracted from Aitchison’s report

Appendix