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Partiality wrecks GADTs' functoriality

How to get Initial Algebra Semantics for GADTs?

TYPES 2023 – València

June 14th, 2023

1. First-order IAS for ADTs

2. Higher-order IAS for ADTs

3. IAS for GADTs?

1. First-order IAS for ADTs

ADTs

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data List (α : Set) : Set where
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data BinTree (α : Set) : Set where
  ∅ : BinTree α
  _⊗_⊗_ : BinTree α → α → BinTree α → BinTree α
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data ℕ : Set where
  zero : ℕ
  succ : ℕ → ℕ
```

Categorical semantics of ADTs

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$\mathcal{C}$: category with finite products, finite sums, and colimits of ω-chains.

To interpret `List A`, take the initial algebra μL_A of:

$$\begin{aligned} L_A &: \mathcal{C} \to \mathcal{C} \\ & X \mapsto 1 + (A \times X) \end{aligned}$$

where A interprets `A`

Categorical semantics of ADTs

ADTs are *uniform* families of inductive types:

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data List (α : Set) : Set where
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Categorical semantics of ADTs

ADTs are *uniform* families of inductive types:

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data List (α : Set) : Set where
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```

$$\mathcal{C} \xrightarrow{L} [\mathcal{C}, \mathcal{C}]_\omega \xrightarrow{\mu} \mathcal{C}$$

$$A \longmapsto L_A \longmapsto \mu L_A$$

2. Higher-order IAS for ADTs

Categorical semantics of ADTs

ADTs can be seen as inductive families of types:

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data List : Set → Set where
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Rework the semantics: to interpret the type constructor `List`, take the initial algebra $\mu\mathcal{L}$ of

$$\mathcal{L} : [\mathcal{C}, \mathcal{C}] \to [\mathcal{C}, \mathcal{C}]$$
$$F \mapsto (X \mapsto 1 + (X \times F(X))$$

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All is well

$\mu\mathcal{L} \simeq \mu \circ L.$

3. IAS for GADTs?

Generalized Algebraic Data Types

GADTs are inductive families of types, with a twist:

```
data Terms : Set → Set where
  nat : ℕ → Terms ℕ
  _,_ : ∀ {α β} → Terms α → Terms β → Terms (α × β)
  π₁  : ∀ {α β} → Terms (α × β) → Terms α
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Naive categorical semantics of GADTs

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$$\mathcal{T} : [\mathcal{C}, \mathcal{C}] \to [\mathcal{C}, \mathcal{C}]$$

$$F \mapsto \left(X \mapsto \quad + \quad + \quad + \quad \right)$$

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Actually, $\mu\mathcal{T} : \mathcal{C} \to \mathcal{C}$ being a functor is already an issue.

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Actually, $\mu\mathcal{T} : \mathcal{C} \to \mathcal{C}$ being a functor is already an issue.

Consider the parity function `p: ℕ → 𝔹`, interpreted by $p : N \to B$. Because of $\mu\mathcal{T}(p) : \mu\mathcal{T}(N) \to \mu\mathcal{T}(B)$, the interpretation of `Terms 𝔹` contains weird elements...

Naive categorical semantics of GADTs

Even more striking with

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- for each $X \in \mathcal{C}$,

$$\exists_X : X \to \mu\mathcal{W}(1),$$

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- for each $X \in \mathcal{C}$,

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- for each $x : 1 \to X$,

$$1 \xrightarrow{\exists_1} \mu\mathcal{W}(1) \xrightarrow{\mu\mathcal{W}(x)} \mu\mathcal{W}(X)$$

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When $\mathcal{C} = \mathbf{Set}$, $\mu\mathcal{W}(X) \neq \emptyset$ whenever $X \neq \emptyset$...

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Issue

$\mu\mathcal{W}(x)$ has to make a new element from $\exists_1$.

Potential solution

Allowing $\mu\mathcal{W}(x)$ to be only partially defined.

Less naive categorical semantics of GADTs

Definition

A structure of partiality on $\mathcal{C}$ is a wide subcategory $\mathcal{D}$ such that

$$\forall f, g \in \mathcal{C}, gf \in \mathcal{D} \implies f \in \mathcal{D}$$

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Example

- $\mathcal{D} = \mathcal{C}$ (trivial)

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- $\mathcal{C} = \mathsf{PSet}$ and $\mathcal{D} = \mathsf{Set}$
- $\mathcal{C} = \mathsf{CPO}$ and $\mathcal{D} = \mathsf{CPO}_{\mathrm{tot}}$

Idea

Interpret the total functions of the language in $\mathcal{D}$ and “spill” in $\mathcal{C}$ for partial functions.

Less naive categorical semantics of GADTs

But...

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- `_≡_` interpreted as: an object $(X \equiv Y)$ for every $X, Y \in \mathcal{C}$
- r interpreted as: a morphism $r_X : 1 \to (X \equiv X)$ in $\mathcal{D}$ for every $X \in \mathcal{C}$

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If $\_ \equiv \_$ extends to a functor $\mathcal{C}^2 \to \mathcal{C}$: for any $x : 1 \to X$,

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If $\_ \equiv \_$ extends to a functor $\mathcal{C}^2 \to \mathcal{C}$: for any $x : 1 \to X$,

$$p_x : \quad 1 \xrightarrow{r_1} 1 \equiv 1 \xrightarrow{(x \equiv \mathrm{id}_1)} (X \equiv 1)$$

Less naive categorical semantics of GADTs

```
trp : ∀ α β → α ≡ β → α → β
trp α α r x = x

trp⁻¹ : ∀ α β → α ≡ β → β → α
trp β β r y = y
```

Less naive categorical semantics of GADTs

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- `trp` interpreted as: a morphism $t_{X,Y} : (X \equiv Y) \times X \to Y$ in $\mathcal{D}$ for every $X, Y \in \mathcal{C}$

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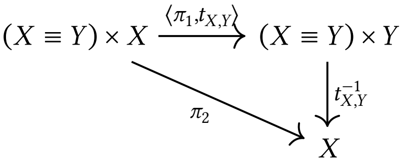
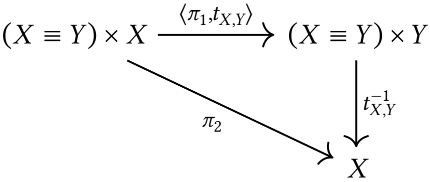
- `(λ p x → trp⁻¹ p (trp p x))` reduces to `(λ p x → x)`:



Less naive categorical semantics of GADTs



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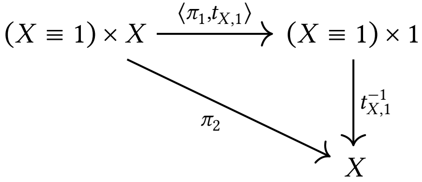

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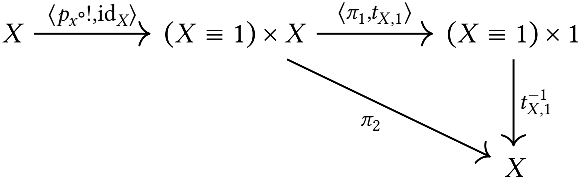

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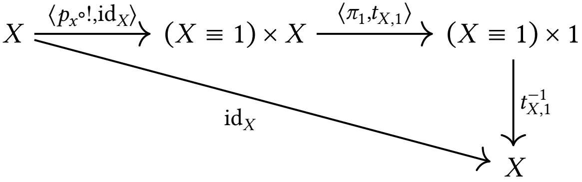

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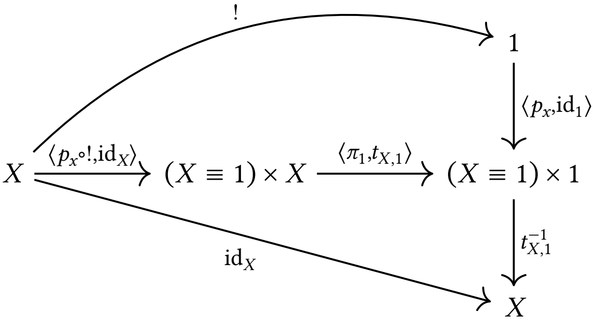

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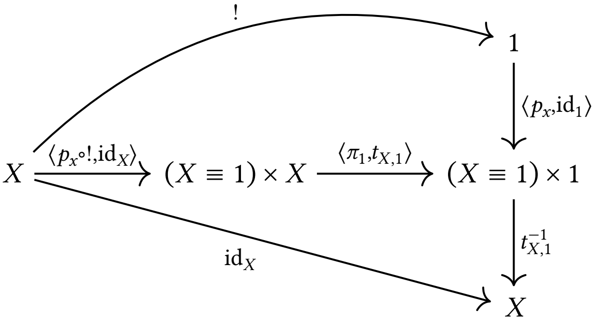
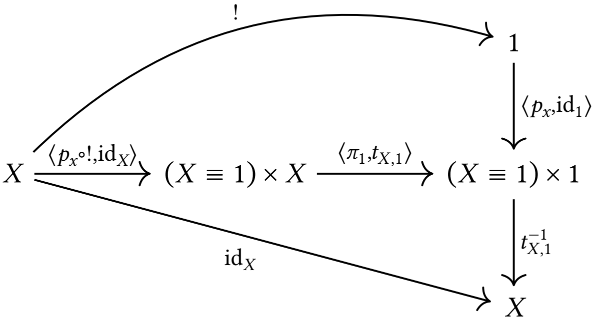


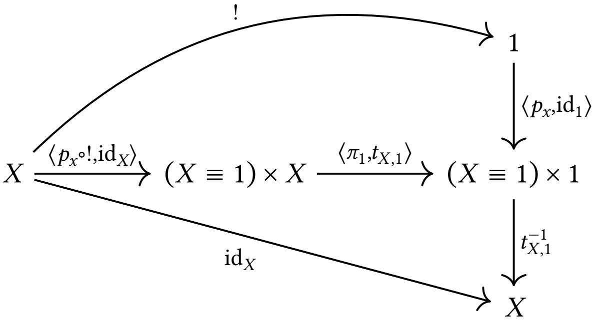
$\Longrightarrow$ X is a retract of 1

Less naive categorical semantics of GADTs



$\Longrightarrow \; X \simeq 1$ whenever $t^{-1}_{X,1}\langle p_x, \mathrm{id}_1\rangle$ in $\mathcal{D}$

Less naive categorical semantics of GADTs



$\implies X \simeq 1$ whenever p_x in $\mathcal{D}$

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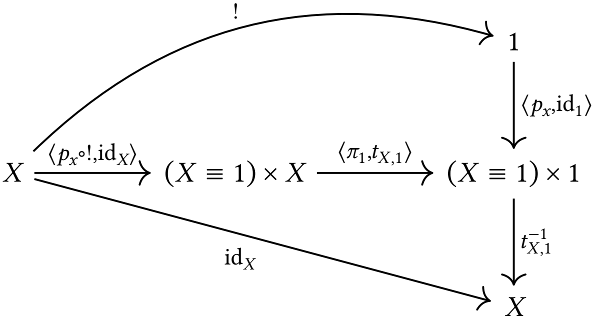

$\Longrightarrow$ $X \simeq 1$ whenever x in $\mathcal{D}$

Less naive categorical semantics of GADTs

Theorem

If GADTs' interpretations extend to functors, the interpretation of any non-empty closed type is trivial.

Thank you.