Self-contained rules for classical and intuitionistic quantifiers

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Natural deduction rules from truth tables

Earlier work: derive natural deduction rules for a connective c from its truth table definition.
Summarizing:

- Generic rule-format, allowing a general proof-theoretic study.
- Produces both the classical and constructive derivation rules for standard connectives (and for less standard connectives).
- Has "good" properties: proof normalization, subformula property, general Kripke semantics (sound and complete), general classical semantics (sound and complete).
- We can study connectives "in isolation", e.g. from the classic rules for $\rightarrow$ one can derive Peirce's Law.
- For monotone connectives (like $\wedge, \vee$), the classical and constructice rules are equivalent; for non-monotonic connectives (like $\rightarrow$, $\neg$) this is not the case.
- One classical non-monotonic connective makes all non-monotonic connectives classical.

Extending with rules for quantifiers

- For any quantifier...ideally...
- but now start from 4 relatively simple ones.
- Derive the rules from the truth table...
- but now I'll just give you the simplified rules.

We want the following.

1. Simple generic rules, preferably extensible to other quantifiers.
2. Intuitionistic rules as a simple variation of the classical ones.
3. Study quantifiers and their rules in isolation.
4. Proof theoretic properties: (Kripke) semantics, soundness and completeness, proof normalization, subformula property.

Four quantifiers

- $\forall x.\varphi$
  The standard $\forall$-rules are intuitionistic. We give classical rules, so one can derive, in intuitionistic logic + classical $\forall$, e.g.
  - $\forall x.(P\,x \vee C) \vdash (\forall x.P\,x) \vee C$
  - $\forall x.\neg\neg\varphi \vdash \neg\neg\forall x.\varphi$ (DNS)
- $\exists x.\varphi$
  The standard $\exists$-rules are intuitionistic. We give classical rules, so one can derive, in intuitionistic logic + classical $\exists$, e.g.
  - $\vdash \exists x.(\exists y.\varphi(y) \to \varphi(x))$.
  - $\neg\neg\exists x.\varphi \vdash \exists x.\neg\neg\varphi$ (DNS for $\exists$)
- $\text{И}x.\varphi$, the "no-quantifier"
  - meaning: "there is no x for which φ holds".
  - $\neg\exists x.\varphi$ and $\text{И}x.\varphi$ and $\forall x.\neg\varphi$ are equivalent, intuitionistically and classically.
- $\text{Ɔ}x.\varphi$, the "counterexample-quantifier"
  - meaning: "there is a x for which φ does not hold".
  - $\exists x.\neg\varphi \vdash \text{Ɔ}x.\varphi$ and $\text{Ɔ}x.\varphi \vdash \neg\forall x.\varphi$, but intuitionistically not the other way around.

The first order language and the formulas

The first order language has:

- an arbitrary finite collection of constants and functions with fixed arity,
- an arbitrary finite collection of predicates with fixed arity,
- witness constants $a_{\forall x.\varphi}$ for all formulas $\varphi(x)$, and similarly $a_{\exists x.\varphi}$, $a_{\text{И}x.\varphi}$ and $a_{\text{Ↄ}x.\varphi}$.

- Classical intuition of a witness constant $a_{\forall x.\varphi}$:
  - if $\forall x.\varphi$ holds, $a_{\forall x.\varphi}$ is an arbitrary element
  - if not $\forall x.\varphi$, $a_{\forall x.\varphi}$ is some element d such that $\neg\varphi[d/x]$.
- Similarly for $a_{\exists x.\varphi}$.
- In the classical semantics, the interpretation of witness constants is exacly that.
- Constructively, the interpretation of witness constants is just the “local fresh parameter” used standardly in deduction rules.

- The judgments are of the form $\Gamma \vdash \varphi$, where all formulas are closed (and may contain the special witness constants).
- In examples, we write trees, with non-discharged hypotheses (from Γ) on top, and φ at the root.
- We have classical rules, indicated with C, and intuitionistic rules, indicated with I. (If nothing is indicated the rules are both.)

Deduction rules for $\forall$, where t is an arbitrary term. We abbreviate $a_{\forall x.\varphi}$ to $a_\forall$.

$$\frac{\vdash \forall x.\varphi}{\vdash \varphi(t)}\ \forall\text{-el} \qquad \Big\| \qquad \frac{\vdash \varphi(a_\forall)}{\vdash \forall x.\varphi}\ \forall\text{-inC} \qquad \frac{\Gamma \vdash \varphi(a_\forall)}{\Gamma \vdash \forall x.\varphi}\ \forall\text{-inI, if } a_\forall \notin \Gamma$$

Example "Drinker's principle"

With classical $\forall$ (and intuitionistic $\to, \exists$):

$$\vdash \exists x.(P\,x \to \forall y.P\,y).$$

We abbreviate $a_\forall := a_{\forall y.P\,y}$.

$$\dfrac{\dfrac{\dfrac{[P\,a_\forall]^1}{\forall y.P\,y}\ {\color{red}\forall\text{-inC}}}{P\,a_\forall \to \forall y.P\,y}\ (1)}{\exists x.(P\,x \to \forall y.P\,y)}$$

With classical $\forall$ (and intuitionistic $\neg$):

$$\forall x.\neg\neg\varphi \vdash \neg\neg\forall x.\varphi.$$

We abbreviate $a_\forall := a_{\forall x.\varphi}$.

$$\dfrac{\dfrac{\dfrac{\forall x.\neg\neg\varphi}{\neg\neg\varphi(a_\forall)} \qquad \dfrac{\dfrac{[\neg\forall x.\varphi]^2 \qquad \dfrac{[\varphi(a_\forall)]^1}{\forall x.\varphi}\ \forall\text{-inC}}{\neg\varphi(a_\forall)}}{\neg\varphi(a_\forall)}\ (1)}{\neg\neg\forall x.\varphi}}{\neg\neg\forall x.\varphi}\ (2)$$

Example “Constant Domain Logic”

With classical $\forall$ (and the rules for $\vee$, which are the same, intuitionistic or classical), we have (x not in C):

$$\forall x.(P\,x \vee C) \vdash (\forall x.P\,x) \vee C.$$

We abbreviate $a_\forall := a_{\forall x.P\,x}$.

$$\dfrac{\dfrac{\forall x.(P\,x \vee C)}{P\,a_\forall \vee C} \qquad \dfrac{\dfrac{[P\,a_\forall]^1}{\forall x.P\,x}\,\color{red}{\forall\text{-inC}}}{(\forall x.P\,x) \vee C} \qquad \dfrac{[C]^1}{(\forall x.P\,x) \vee C}}{(\forall x.P\,x) \vee C}\,(1)$$

It is known that this axiom scheme is complete for Kripke models with constant domains.

With classical ∀ (and the intuitionistic rules for ∨, ¬, ∃), we have:

$$\forall x.(P\,x \vee \neg P\,x), \neg\forall x.P\,x \vdash \exists x.\neg P\,x.$$

We abbreviate $a_\forall := a_{\forall x.P\,x}$.

$$\dfrac{\dfrac{\forall x.(P\,x \vee \neg P\,x)}{P\,a_\forall \vee \neg P\,a_\forall} \qquad \dfrac{\neg\forall x.P\,x \qquad \dfrac{[P\,a_\forall]^1}{\forall x.P\,x}\,\forall\text{-inC}}{\exists x.\neg P\,x} \qquad \dfrac{[\neg P\,a_\forall]^1}{\exists x.\neg P\,x}}{\exists x.\neg P\,x}\,(1)$$

It was already known that MP follows from the axiom scheme for CDL: $\forall x.(P\,x \vee C) \vdash (\forall x.P\,x) \vee C$.

The derivation rules for $\exists$

Deduction rules for $\exists$, where t is an arbitrary term. We abbreviate $a_{\exists x.\varphi}$ to $a_\exists$.

$$\frac{\vdash \exists x.\varphi}{\vdash \varphi(a_\exists)}\ \exists\text{-elC} \qquad \frac{\Gamma \vdash \exists x.\varphi \quad \Gamma, \varphi(a_\exists) \vdash \psi}{\Gamma \vdash \psi}\ \exists\text{-elI}^{(*)} \quad \Bigg\| \quad \frac{\vdash \varphi(t)}{\vdash \exists x.\varphi}\ \exists\text{-in}$$

(*) if $a_\exists \notin \Gamma, \psi$

With classical $\exists$ and intuitionistic $\to$ we can show the "Existence Principle": $\vdash \exists x.(\exists y.P\,y) \to P\,x$. We abbreviate $a_\exists := a_{\exists y.P\,y}$.

$$\dfrac{\dfrac{\dfrac{[\exists y.P\,y]^1}{P\,a_\exists}\ \exists\text{-elC}}{(\exists y.P\,y) \to P\,a_\exists}\ (1)}{\exists x.(\exists y.P\,y) \to P\,x}$$

Recall $\text{И}x.\varphi$ says "there is no x for which φ holds".
Here t is an arbitrary term. We abbreviate $a_{\text{И}x.\varphi}$ to $a_{\text{И}}$.

$$\frac{\vdash \text{И}x.\varphi \quad \vdash \varphi(t)}{\vdash \psi}\ \text{И-el}$$

$$\frac{\text{И}x.\varphi \vdash \psi \quad \varphi(a_{\text{И}}) \vdash \psi}{\vdash \psi}\ \text{И-inC} \qquad \frac{\Gamma, \varphi(a_{\text{И}}) \vdash \text{И}x.\varphi}{\Gamma \vdash \text{И}x.\varphi}\ \text{И-inI, if } a_{\text{И}} \notin \Gamma$$

The classical interpretation of $a_{\text{И}x.\varphi}$ is:

$$a_{\text{И}x.\varphi} = \begin{cases} \text{an arbitrary element of } D & \text{if } \text{И}x.\varphi \\ \text{some element } d \text{ for which } \varphi(d) & \text{if not } \text{И}x.\varphi. \end{cases}$$

So we will have $\varphi(a_{\text{И}x.\varphi}) \Longleftrightarrow \neg\text{И}x.\varphi$.

Note: in case x doesn't occur in φ, the formula $\text{И}x.\varphi$ is just $\neg\varphi$ and the rules are just the rules for negation.

The derivation rules for Ↄ

Recall $Ↄx.\varphi$ says "there is an x for which φ does not hold'. Here t is an arbitrary term. We abbreviate $a_{Ↄx.\varphi}$ to $a_Ↄ$

$$\frac{\vdash Ↄx.\varphi \quad \vdash \varphi(a_Ↄ)}{\vdash \psi}\,Ↄ\text{-elC} \qquad \frac{\Gamma \vdash Ↄx.\varphi \quad \Gamma \vdash \varphi(a_Ↄ)}{\Gamma \vdash \psi}\,Ↄ\text{-elI}^{(*)}$$

$$\frac{Ↄx.\varphi \vdash \psi \quad \varphi(t) \vdash \psi}{\vdash \psi}\,Ↄ\text{-inC} \qquad \frac{\varphi(t) \vdash Ↄx.\varphi}{\vdash Ↄx.\varphi}\,Ↄ\text{-inI}$$

(*) if $a_Ↄ \notin \Gamma$

The classical interpretation of $a_{Ↄx.\varphi}$ is:

$$a_{Ↄx.\varphi} = \begin{cases} \text{some element } d \text{ for which not } \varphi(d) & \text{if } Ↄx.\varphi \\ \text{an arbitrary element of } D & \text{if not } Ↄx.\varphi. \end{cases}$$

So we will have $\varphi(a_{Ↄx.\varphi}) \iff \neg Ↄx.\varphi$.

Note: again, if $x \notin \varphi$, we find that $Ↄx.\varphi$ is just $\neg\varphi$.

What is $\text{Ↄ}x.\varphi$ intuitionistically?

Recall $\text{Ↄ}x.\varphi$ says "there is an x for which φ does not hold'.

$$\dfrac{\Gamma \vdash \text{Ↄ}x.\varphi \qquad \Gamma \vdash \varphi(a_{\text{Ↄ}})}{\Gamma \vdash \psi}\ \text{Ↄ-elI}^{(*)} \qquad \dfrac{\varphi(t) \vdash \text{Ↄ}x.\varphi}{\vdash \text{Ↄ}x.\varphi}\ \text{Ↄ-inI}$$

$\exists x.\neg\varphi \vdash \text{Ↄ}x.\varphi$ and $\text{Ↄ}x.\varphi \vdash \neg\forall x.\varphi$ (but intuitionistically not the other way around).

$$\dfrac{\exists x.\neg\varphi \qquad \dfrac{\dfrac{[\neg\varphi(a_\exists)]^2 \qquad [\varphi(a_\exists)]^1}{\text{Ↄ}x.\varphi}}{\text{Ↄ}x.\varphi}\ \text{Ↄ-inI}\,(1)}{\text{Ↄ}x.\varphi}\ \exists\text{-inI}\,(2) \qquad \dfrac{\dfrac{\text{Ↄ}x.\varphi \qquad \dfrac{[\forall x.\varphi]^1}{\varphi(a_{\text{Ↄ}})}}{\neg\forall x.\varphi}\ \text{Ↄ-elI}}{\neg\forall x.\varphi}\ (1)$$

The Kripke semantics of $\reflectbox{E}x.\varphi$

- $\reflectbox{E}x.\varphi$ is really in between $\exists x.\neg\varphi$ and $\neg\forall x.\varphi$.
- We can make simple Kripke counter models to $\reflectbox{E}x.\varphi \vdash \exists x.\neg\varphi$ and to $\neg\forall x.\varphi \vdash \reflectbox{E}x.\varphi$.

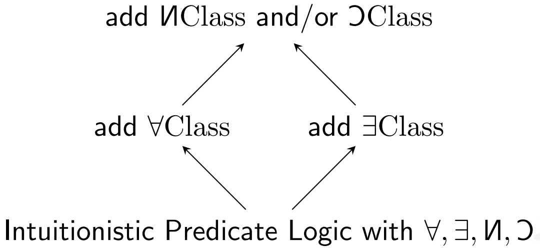
$$w \Vdash \exists x.\neg\varphi \quad \Leftrightarrow \quad \exists d \in D(w)\, \forall w' \geq w (w' \nVdash \varphi(d))$$

$$w \Vdash \reflectbox{E}x.\varphi \quad \Leftrightarrow \quad \forall w' \geq w\, \exists d \in D(w')(w' \nVdash \varphi(d))$$

$$w \Vdash \neg\forall x.\varphi \quad \Leftrightarrow \quad \forall w' \geq w\, \exists w'' \geq w'\, \exists d \in D(w'')(w'' \nVdash \varphi(d)).$$

The classical rules for ∀, ∃ do not make the logic classical

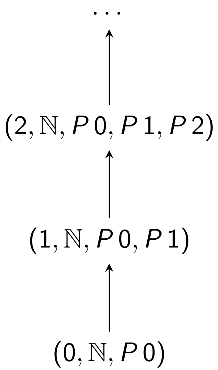
We have the following inclusions, where the top is full classical logic.



- Does ∀Class (or ∃Class) make the logic fully classical?
  Answer: no, in Kripke models with a constant singleton domain the classical rules for ∀ and ∃ are true.
- Are the ∀Class and ∃Class rules derivable from eachother?
  Answer: no, there are Kripke models where one is true and the other not and vice versa.

A Kripke model where $\exists$Class holds and $\forall$Class not

$\mathcal{K} := \langle \mathbb{N}, \leq, \mathbb{N}, I, \mathsf{At}\rangle$, so domain $\mathbb{N}$ for all worlds $w \in \mathbb{N}$, and $\leq$ is the standard ordering on $\mathbb{N}$. $\mathsf{At}(w) := \{P\,n \mid n \leq w\}$



$0 \Vdash \exists x.(P\,x \to \forall y.P\,y)$ only if there is an $n \in \mathbb{N}$ such that $\forall w \in \mathbb{N}(w \Vdash P\,n \Rightarrow w \Vdash \forall y.P\,y)$. Now $w \nVdash \forall y.P\,y$ for all w, but for every $n \in \mathbb{N}$ there is a w such that $w \Vdash P\,n$. So $0 \nVdash \exists x.(P\,x \to \forall y.P\,y)$.

A more precise semantic charachterisation

Proposition

$\forall\mathrm{Class}$ holds in all Kripke models $\mathcal{K} := \langle W, \leq, D, I, \mathsf{At}\rangle$ where

- $D(w)$ is a fixed domain D for all $w \in W$,
- $\leq$ is a total order
- all subsets S of W have a **largest** element.

$\exists\mathrm{Class}$ holds in all Kripke models $\mathcal{K} := \langle W, \leq, D, I, \mathsf{At}\rangle$ where

- $D(w)$ is a fixed domain D for all $w \in W$,
- $\leq$ is a total order
- all subsets S of W have a **smallest** element.

A more precise logical charachterisation

Proposition

1. Classical $\forall$ is equivalent to the Drinker's Principle.
   Given Γ, ψ that do not contain witness constants, we have

   $\Gamma \vdash \psi$ with classical $\forall$ in intuitionistic $\to\exists$-logic

   $\Longleftrightarrow$

   $\Gamma \vdash \psi$ with axiom scheme $\vdash \exists y.(\varphi(y) \to \forall x.\varphi)$
   in intuitionistic $\to\exists\forall$-logic

2. Classical $\exists$ is equivalent to the Existence Principle. That is, given Γ, ψ that do not contain witness constants, we have

   $\Gamma \vdash \psi$ with classical $\exists$ in intuitionistic $\to$-logic

   $\Longleftrightarrow$

   $\Gamma \vdash \psi$ with axiom scheme $\vdash \exists y.(\exists x.\varphi) \to \varphi(y)$
   in intuitionistic $\to\exists$-logic

Overview of some classically derivable ∀∃-statements

$$
\begin{array}{rcll}
\forall x.\neg\neg\varphi & \vdash & \neg\neg\forall x.\varphi & \text{DNS} \\
\neg\neg\exists x.\varphi & \vdash & \exists x.\neg\neg\varphi & \text{DNS for } \exists \\
\forall x.(\varphi \vee C) & \vdash & (\forall x.\varphi) \vee C & \text{CDL} \\
C \to \exists x.\varphi & \vdash & \exists x.(C \to \varphi) &
\end{array}
$$

for x not in C.

Conclusion and further work

Summarizing:

- We have "stand alone" natural deduction rules for classical (and intuitionistic) predicate logic $\forall, \exists, И, Ɔ$.
- The intuitionistic rules are a variation on the classical ones.
- Kripke semantics that is sound (and completeness to be checked in detail).
- Classical semantics that is sound and complete.
- Derivations satisfying the subformula property for a number of well-known classically provable statements.
- The rules follow mostly the "standard form", so proof normalizations should work.

Further work:

- Characterise the precise fragment intuitionistic proposition logic + classical $\forall$ (or $\exists$).
- Compare with other known logics extending intuitionistic logic.
- Extend to other quantifiers.
- Proof term interpretation and proof of normalization

Questions?