Diller-Nahm Bar Recursion

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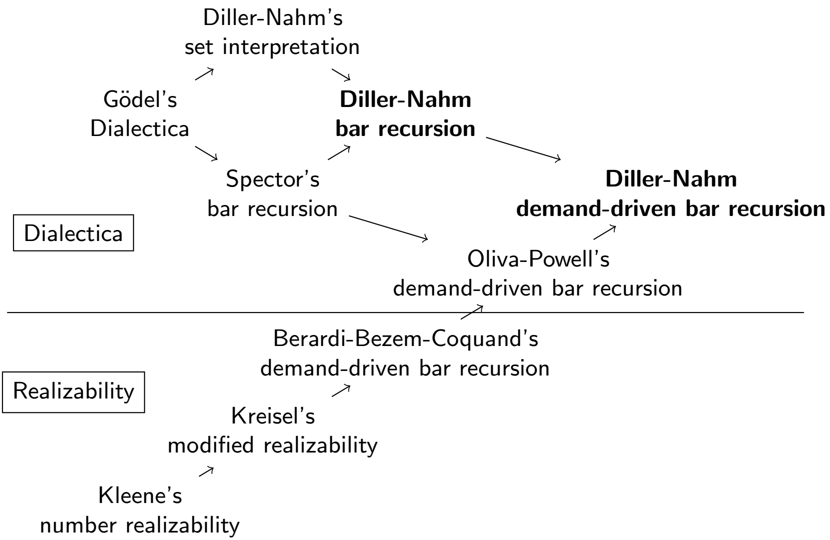
A bit of history: computational interpretations

- Interpretations of arithmetic
  - 1941: Gödel's Dialectica interpretation (published in 1958)
  - 1945: Kleene's number realizability
  - 1959: Kreisel's modified realizability
  - 1974: Diller-Nahm's set-based variant of Dialectica

- Extension to analysis via bar recursion
  - 1962: Spector's bar recursion for Dialectica
  - 1998: Berardi-Bezem-Coquand's demand-driven bar recursion for Kreisel's realizability
  - 2017: Oliva-Powell's demand-driven bar recursion for Dialectica

Our contributions



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in $\overset{\bigcirc}{A}$, $\bigcirc$ represents witnesses of A

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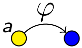

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input **output**

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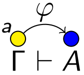

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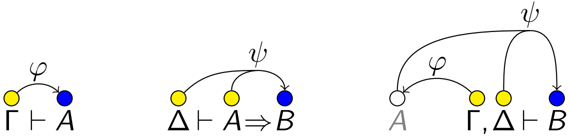

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in $\underset{\square}{\overset{\bigcirc}{A}}$, $\bigcirc/\square$ represent witnesses/counter-witnesses of A

$A_D\,(a \,\|\, b)$ (if a is a witness of A and b is a counter-witness of A)
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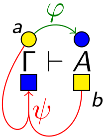

such that if $\Gamma_D\,(a \parallel \psi\,(a, b))$ then $A_D\,(\varphi\,(a) \parallel b)$

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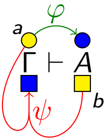

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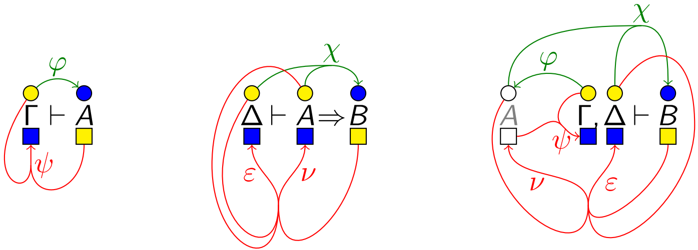
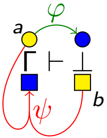


Dialectica: negation



such that if $\Gamma_D \left(a \,\|\, \psi\left(a, b\right) \right)$ then $\bot_D \left(\varphi\left(a\right) \,\|\, b \right)$

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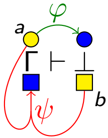
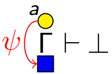


such that if $\Gamma_D\,(a \,\|\, \psi\,(a, b))$ then $\bot_D\,(\varphi\,(a) \,\|\, b)$

but $\bot_D\,(\_ \,\|\, \_)$ is false
and witnesses and counter-witnesses of $\bot$ are meaningless



such that **not** $\Gamma_D\,(a \,\|\, \psi\,(a))$

Double-Negation Shift

$$\forall x \neg\neg A \Rightarrow \neg\neg\forall x\, A$$

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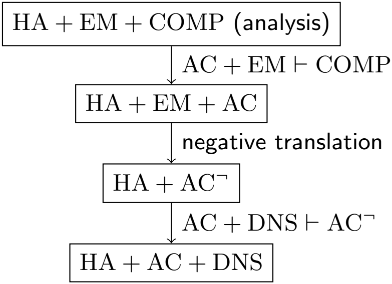
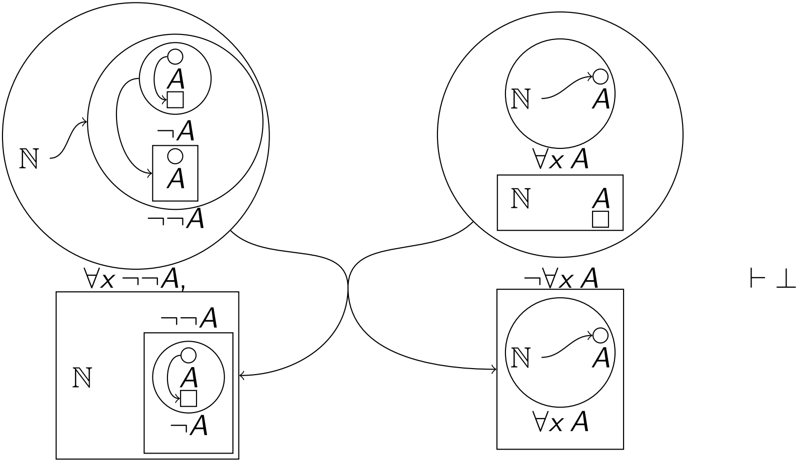


HA: Heyting Arithmetic (intuitionistic)
COMP: Comprehension Axiom
EM: Excluded Middle
AC: Axiom of Choice
DNS: Double-Negation Shift

computational interpretation of DNS

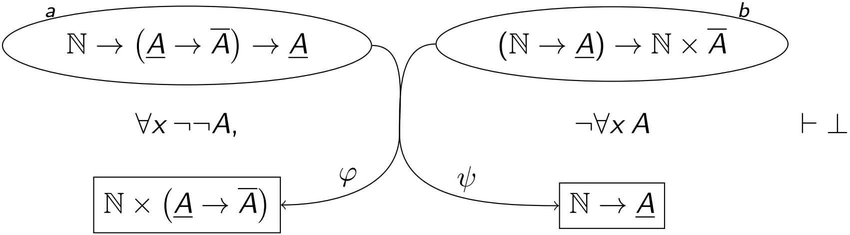
$\rightsquigarrow$ computational interpretation of analysis

Dialectica interpretation of DNS



Dialectica interpretation of DNS

$\underline{A}$: witnesses of A $\quad\quad$ $\overline{A}$: counter-witnesses of A



$$\varphi' = \varphi\,(a, b) \qquad\qquad \psi' = \psi\,(a, b)$$

such that if [some condition on φ']

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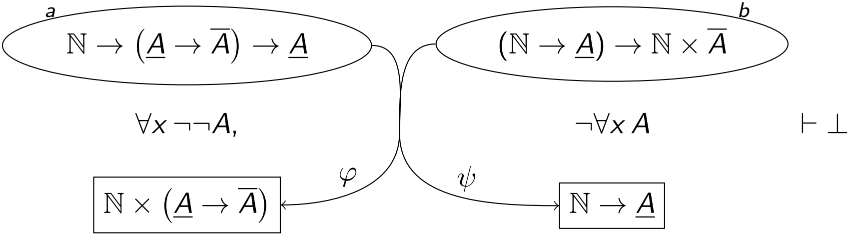

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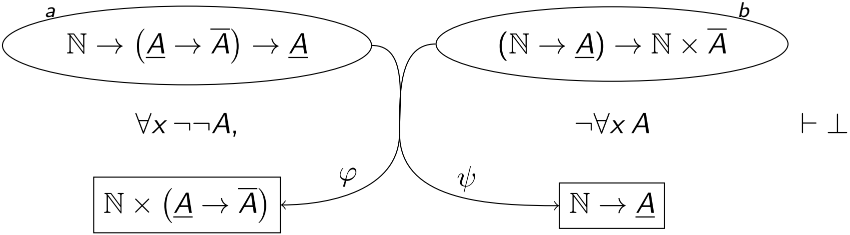

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that is, ψ' wins against $b(\psi')$ on game $\forall x A$

that is, $\psi'(n)$ wins against c on game $A(n)$
where $b(\psi') = (n, c)$

Complete approximations, correct sequences

if $b : (\mathbb{N} \to \underline{A}) \to \mathbb{N} \times \overline{A}$

- $[a_0, \ldots, a_{m-1}]$ is a **complete approximation** if $b_1\,(a_0, \ldots, a_{m-1}, 0, \ldots, 0, \ldots) < m$
- $\alpha : \mathbb{N} \to \underline{A}$ is a **correct sequence** if $\alpha\,(n)$ wins against c on $A\,(n)$, where $b\,(\alpha) = (n, c)$

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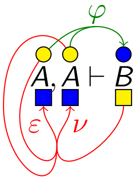
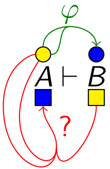
ψ' is a correct sequence built via successive approximations

$$\mathit{bar\ rec}\,[a_0, \ldots, a_{m-1}] = \begin{cases} [a_0, \ldots, a_{m-1}] \\ \qquad \text{if } [a_0, \ldots, a_{m-1}] \text{ complete} \\ \mathit{bar\ rec}\,[a_0, \ldots, a_{m-1}, a] \\ \qquad \text{for some well-chosen } a \text{ otherwise} \end{cases}$$

$$\psi' = (\mathit{bar\ rec}\,[\,])\,, 0, \ldots, 0, \ldots$$

Dialectica: the contraction problem





Gödel's Dialectica: play the game and keep the winner
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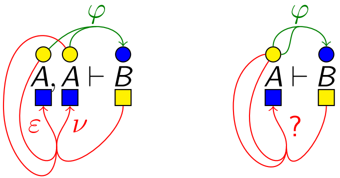

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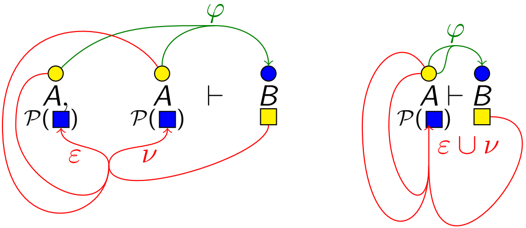
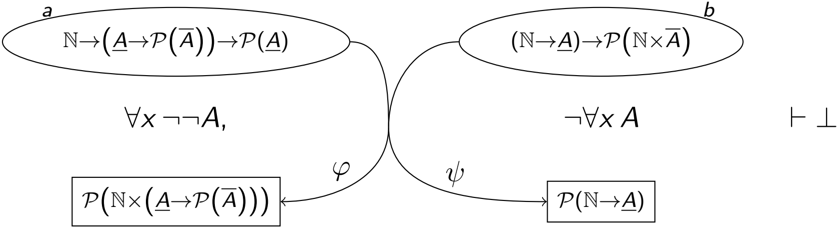
Diller-Nahm variant: catch 'em all!



Diller-Nahm interpretation of DNS



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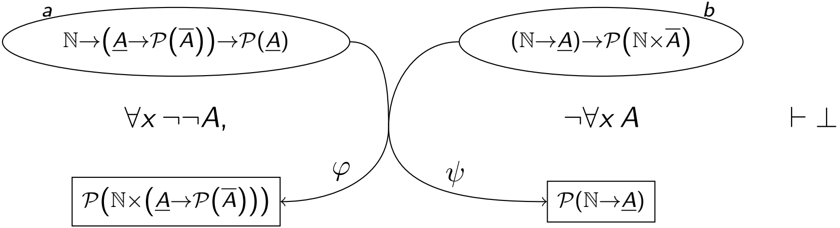

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that is, $\exists \alpha \in \psi'$ such that $\forall (n, c) \in b(\alpha)$
$\alpha(n)$ wins against c on game $A(n)$

Complete approximations, correct sequences, revisited

if $b : (\mathbb{N} \to \underline{A}) \to \mathcal{P}\left(\mathbb{N} \times \overline{A}\right)$

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$$\psi' = \{a_0, \dots, a_{m-1}, 0, \dots, 0, \dots \mid [a_0, \dots, a_{m-1}] \in bar\ rec\,[]\}$$

only one sequence of ψ' has to be correct

Demand-driven bar recursion

Until now we built approximations of the form:

n	0	1	$\dots$	$m-1$	m	$\dots$	$\dots$
$\alpha(n)$	a_0	a_1	$\dots$	a_{m-1}	?	?	?

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Bar recursion was extended to arbitrary approximations of the following form, first in the context of realizability and more recently in the context of Dialectica:

n	0	1	$\dots$	m_1	$\dots$	m_2	$\dots$	m_3	$\dots$
$\alpha(n)$	?	a_1	?	a_{m_1}	?	a_{m_2}	?	a_{m_3}	?

The technique shown before in the Diller-Nahm setting extends to demand-driven bar recursion.

Final remarks

There are many technicalities:

- Extensions are computed via a complex interaction with φ
- Termination of bar recursion is far from being obvious
- Diller-Nahm interpretation requires an implementation of finite sets
- ...

In this talk I put all this under the carpet, trying to give general ideas.

If you're interested, details are in the associated FSCD paper (available on my webpage).

thank you