Towards an Interpretation of Inaccessible Sets in Martin-Löf Type Theory with One Mahlo Universe

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Background: Ordinal Analysis

- Ordinal analysis is an area in proof theory of mathematical logic
- This area aims to measure the strength of a proof system S by using the order type of a primitive recursive well ordering
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- The analysis has been given to:
  - first-order arithmetic
  - subsystems of second-order arithmetic
  - Kripke-Platek set theory
  - Martin-Löf type theory
  - $\cdots$

Background: Related Work in Ordinal Analysis

- Related work:

Setzer [1, 2]	MLTT with one universe
Rathjen-Griffor-Palmgren [3]	MLTT with the types M and Q $(\mathbf{MLQ})$
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Argument in Rathjen-Griffor-Palmgren [3]

- formulates the extension $\mathbf{CZF}_{\pi}$ of Aczel's $\mathbf{CZF}$ with the existence of Mahlo's inaccessible sets of all transfinite orders [6]
- interprets $\mathbf{CZF}_{\pi}$ in $\mathbf{MLQ}$ by extending Aczel's interpretation of $\mathbf{CZF}$ in MLTT [7, 8, 9]

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This Talk's Question

Is $\mathbf{CZF}_\pi$ interpretable in $\mathbf{MLM}$?

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  Understanding what set-construction is possible in a Mahlo universe
  - Aczel [7, 8, 9] showed that
    the basic set-construction in $\mathbf{CZF}$ are interpretable in MLTT
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This Talk's Question in Another Form

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- We show that the type-theoretic counterpart $\mathcal{V}^{\alpha}_{(a,f)}$ of α-set-inaccessible sets in [3] can be defined in $\mathbf{MLM}$
  - Informally, an α-set-inaccessible set γ is inaccessible from β-set-inaccessible sets for any set β in the transitive closure of the set α

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- In $\mathbf{MLQ}$, $\mathcal{V}^{\alpha}_{(a,f)}$ was constructed by using the two types M and Q
  - Q is an inductive type of codes for operators which gives universes closed under universe operators constructed previously
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- Replace M in $\mathbf{MLQ}$ with the Mahlo universe V in $\mathbf{MLM}$
- Formulate a higher-order universe operator $\mathrm{u}^{\mathbb{M}}$ which emulates Q
  - Use the reflection property of the Mahlo universe V to define $\mathrm{u}^{\mathbb{M}}$

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  - Construction of a function f on $\Sigma_{(x:\mathrm{V})}(\mathrm{T}_{\mathrm{V}} x \to \mathrm{V})$ can be done in a subuniverse $\widehat{\mathrm{U}}_f$ of V

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$\widehat{\mathrm{U}}_f$ is a subuniverse of V with the decoding function $\widehat{\mathrm{T}}_f : \mathrm{U}_f \to \mathrm{V}$ such that $(\widehat{\mathrm{U}}_f, \widehat{\mathrm{T}}_f)$ is closed under f

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  - res^0_f and res^1_f provide the restriction of f to U_f

$$\lambda y.(\mathrm{res}^0_f\ y, \mathrm{res}^1_f\ y) : \Sigma_{(x:\mathrm{U}_f)}(\mathrm{T}_f x \to \mathrm{U}_f) \to \Sigma_{(x:\mathrm{U}_f)}(\mathrm{T}_f x \to \mathrm{U}_f)$$

An Example of Subuniverses of a Mahlo Universe

Super Universe

- By reflecting $z : \Sigma_{(x:\mathrm{V})}(\mathrm{T_V} x \to \mathrm{V}) \vdash \lambda y.z : \Sigma_{(x:\mathrm{V})}(\mathrm{T_V} x \to \mathrm{V}) \to \Sigma_{(x:\mathrm{V})}(\mathrm{T_V} x \to \mathrm{V})$, we have $(\widehat{\mathrm{U}}_{f_1}, \widehat{\mathrm{T}}_{f_1})$ with $f_1 := \lambda y.z$
- Put $g := \lambda z.(\widehat{\mathrm{U}}_{f_1}, \widehat{\mathrm{T}}_{f_1}) : \Sigma_{(x:\mathrm{V})}(\mathrm{T_V} x \to \mathrm{V}) \to \Sigma_{(x:\mathrm{V})}(\mathrm{T_V} x \to \mathrm{V})$ then we obtain $(\widehat{\mathrm{U}}_g, \widehat{\mathrm{T}}_g)$ by reflection again
- $(\widehat{\mathrm{U}}_g, \widehat{\mathrm{T}}_g)$ is a super universe because g is a universe operator

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- Put $\mathrm{Fam}(\mathrm{V}) := \Sigma_{(x:\mathrm{V})}(\mathrm{T_V}x \to \mathrm{V})$
- the type O of first-order operators, and
  the type $\mathrm{Fam}(\mathrm{O})$ of families of first-order operators:

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3. Iterate $\mathrm{u}^{\mathbb{M}}$ along a given iterative set $\alpha : \mathrm{W}_{(x:\mathrm{V})}\mathrm{T_V}x$ by using transfinite induction on the transitive closure of α

Step 1: Composing Operators into One

- For any $(z, v) : \mathrm{Fam}(\mathrm{O})$, we have

$$v\,y_1 : \mathrm{O} \text{ with } y_1 : \mathrm{T_V}z$$
$$v\,y_2 : \mathrm{O} \text{ with } y_2 : \mathrm{T_V}z$$
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- Compose these operators into an operator $f^{\mathbb{M}} : \mathrm{O}$
(the details are omitted here)
  - $f^{\mathbb{M}}$ simulates $v\, y : \mathrm{O}$ for each $y : \mathrm{T_V}\, z$
  - In addition, we make it possible for $f^{\mathbb{M}}$ to simulate a given $(x, y) : \mathrm{Fam}(\mathrm{V})$
- So $f^{\mathbb{M}}$ has (z, v) and (x, y) as parameters, and can be written as $f^{\mathbb{M}}[z, v, x, y]$

Step 2: Higher-Order Universe Operator $\mathrm{u}^{\mathbb{M}}$

- Since $f^{\mathbb{M}}[z, v, x, y]$ is of type O, we are able to reflect it in V

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$\widehat{\mathrm{U}}_{f^{\mathbb{M}}[z,v,x,y]}$ is a subuniverse of V such that

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- We then define $\mathrm{u}^{\mathbb{M}} : \mathrm{Fam}(\mathrm{O}) \to \mathrm{O}$ as

$$\mathrm{u}^{\mathbb{M}}\ (z, v)\ (x, y) := (\widehat{\mathrm{U}}_{f^{\mathbb{M}}[z,v,x,y]}, \widehat{\mathrm{T}}_{f^{\mathbb{M}}[z,v,x,y]})$$

- $\mathrm{u}^{\mathbb{M}}$ is a higher-order universe operator in Palmgren's sense [12] because it takes a family of first-order operators as an argument

Step 3: Iterating $\mathrm{u}^{\mathbb{M}}$ Transfinitely

- The type $\mathbf{V}$ of Aczel's iterative sets is defined as $\mathbf{V} := \mathrm{W}_{(x:\mathrm{V})}\mathrm{T}_{\mathrm{V}}x$ with $\mathrm{index} : \mathbf{V} \to \mathrm{V}$ and $\mathrm{pred} : \Pi_{(x:\mathbf{V})}\mathrm{T}_{\mathrm{V}}(\mathrm{index}\, x) \to \mathbf{V}$ s.t. $\mathrm{index}\,(\mathrm{sup}\ a\ f) = a$ and $\mathrm{pred}\,(\mathrm{sup}\ a\ f) = f$
- We also define the transitive closure α_{tc} of α for each $\alpha : \mathbf{V}$

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Transfinite Induction on α_{tc}

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  - This gives an α-th subuniverse $\mathcal{M}^{\alpha}_{(a,f)}$ being closed under each universe operators obtained by iterating $\mathrm{u}^{\mathbb{M}}$ up to β for all $\beta \in \alpha_{\mathrm{tc}}$
- Let $\mathrm{T}^{\alpha}_{(a,f)}$ be the decoding function of $\mathcal{M}^{\alpha}_{(a,f)}$, then define the type $\mathcal{V}^{\alpha}_{(a,f)}$ as

$$\mathcal{V}^{\alpha}_{(a,f)} := \mathrm{W}_{(x:\mathcal{M}^{\alpha}_{(a,f)})}\mathrm{T}^{\alpha}_{(a,f)}x$$

Concluding Remarks

- We have defined the type-theoretic counterpart $\mathcal{V}^{\alpha}_{(a,f)}$ of α-set-inaccessible sets by using one Mahlo universe
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  - See our construction from the viewpoint of recent type-theoretic approaches to ordinals in the context of homotopy type theory [13, 14]

Thank you for your attention!

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Definition of $\mathrm{u}^{\mathbb{M}}$

Definition (Higher-Order Universe Operator $\mathrm{u}^{\mathbb{M}}$)

Let $(z, v) : \mathrm{Fam}(\mathrm{O})$ and $(x, y) : \mathrm{Fam}(\mathrm{V})$ be given. Define a function

$$h : \Pi_{(w:\mathrm{Fam}(\mathrm{V}))}\Big(((\mathrm{N}_1 + \mathrm{T}_{\mathrm{V}}x) + \mathrm{T}_{\mathrm{V}}z) + \Sigma_{(w':\mathrm{T}_{\mathrm{V}}z)}\mathrm{T}_{\mathrm{V}}\mathrm{p}_1(v\, w'\, w) \to \mathrm{V}\Big)$$

by

$$h\, w\, (\mathrm{i}(\mathrm{i}(\mathrm{i}\, x_1))) = x \text{ with } x_1 : \mathrm{N}_1, \tag{1}$$

$$h\, w\, (\mathrm{i}(\mathrm{i}(\mathrm{j}\, x_2))) = y\, x_2 \text{ with } x_2 : \mathrm{T}_{\mathrm{V}}x, \tag{2}$$

$$h\, w\, (\mathrm{i}(\mathrm{j}\, y_1)) = \mathrm{p}_1(v\, y_1\, w) \text{ with } y_1 : \mathrm{T}_{\mathrm{V}}z, \tag{3}$$

$$h\, w\, (\mathrm{j}\, (y_1, z_1)) = \mathrm{p}_2(v\, y_1\, w)\, z_1 \text{ with } y_1 : \mathrm{T}_{\mathrm{V}}z \text{ and } z_1 : \mathrm{T}_{\mathrm{V}}\mathrm{p}_1(v\, y_1\, w). \tag{4}$$

Put $f^{\mathbb{M}}[z, v, x, y] := \lambda w.(((\widehat{\mathrm{N}_1}_{\mathrm{V}}\ \widehat{+}_{\mathrm{V}}\ x)\ \widehat{+}_{\mathrm{V}}\ z)\ \widehat{+}_{\mathrm{V}}\ \widehat{\Sigma}_{\mathrm{V}}(z, (w')\mathrm{p}_1(v\, w'\, w)),\ h\, w)$.
We then define $\mathrm{u}^{\mathbb{M}} : \mathrm{Fam}(\mathrm{O}) \to \mathrm{O}$ as
$\mathrm{u}^{\mathbb{M}}\, (z, v)\, (x, y) := (\widehat{\mathrm{U}}_{f^{\mathbb{M}}[z,v,x,y]}, \widehat{\mathrm{T}}_{f^{\mathbb{M}}[z,v,x,y]})$ by reflecting $f^{\mathbb{M}}[z, v, x, y]$ in V.

Definition of $\mathcal{V}^{\alpha}_{(a,f)}$

- The transfinite induction tcTI on the transitive closure α_{tc} of $\alpha : \mathbf{V}$ is definable:

$$\mathrm{tcTI} : \Pi_{(\alpha:\mathbf{V})}(\Pi_{(x:\mathrm{T_V}(\mathrm{index}\,\alpha_{\mathrm{tc}}))} F\,(\mathrm{pred}\,\alpha_{\mathrm{tc}}\,x) \to F\,\alpha) \to \Pi_{(\alpha:\mathbf{V})} F\,\alpha$$

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Definition (The Type $\mathcal{V}^{\alpha}_{(a,f)}$ of α-Iterative Sets)

Put $\Phi : \mathbf{V} \to \mathrm{O}$ as

$$\Phi\ \alpha = \mathrm{tcTI}\ (\lambda\beta.\lambda x.\mathrm{u}^{\mathbb{M}}\ (\mathrm{index}\ \beta_{\mathrm{tc}}, x))\ \alpha.$$

For any $a : \mathrm{V}$ and $f : \mathrm{T_V} a \to \mathrm{V}$, we define the α-th subuniverse $\mathcal{M}^{\alpha}_{(a,f)}$ of V and the type $\mathcal{V}^{\alpha}_{(a,f)}$ of α-th iterative sets on $\mathcal{M}^{\alpha}_{(a,f)}$ as follows:

$$\mathcal{M}^{\alpha}_{(a,f)} := \mathrm{T_V}(\mathrm{p}_1(\Phi\ \alpha\ (a,f)))$$
$$\mathrm{T}^{\alpha}_{(a,f)} := \lambda x.\mathrm{T_V}(\mathrm{p}_2(\Phi\ \alpha\ (a,f))\ x)$$
$$\mathcal{V}^{\alpha}_{(a,f)} := \mathrm{W}_{(x:\mathcal{M}^{\alpha}_{(a,f)})}\mathrm{T}^{\alpha}_{(a,f)} x$$