Categories as Semicategories with Identities

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Motivation

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Develop ∞-category theory internally to HoTT.

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- Describe the natural higher structure of the Universes internally.
- Develop a syntactic theory of higher categories.
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Very difficult problem due to coherence issues.

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How?

1. Define the composition structure $\rightsquigarrow$ ∞-semicategory.
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The type of identity-structure should be a proposition:

- An ∞-semicategory should be an ∞-category in only one way.
- Define ∞-categories as a sub-type of ∞-semicategories.

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Generalisation:

In this talk we will focus on wild categories:

- No higher coherence conditions.
- No truncation conditions.

$\rightsquigarrow$ Wild categories generalise ∞-categories.

Outline

1. Wild semicategory

A wild semicategory is a tuple $(\mathrm{Ob}, \mathrm{hom}, \circ, \alpha)$ consisting of:

- $\mathrm{Ob} : U$.
- $\mathrm{hom} : \mathrm{Ob} \to \mathrm{Ob} \to U$.
- $\_ \circ \_ : \prod_{x,y,z:\mathrm{Ob}} \mathrm{hom}\, y\ z \to \mathrm{hom}\, x\ y \to \mathrm{hom}\, x\ z$.
- $\alpha : \prod_{f,g,h} (h \circ g) \circ f = h \circ (g \circ f)$.

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Goal of the talk:
Define an identity structure for a wild semicategory.

Preparation

We can still define notions of equivalence and neutrality using the type theoretic equivalences and identity types:

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For x, y : Ob,

$$\mathrm{eqv}(x,y) :\equiv \sum_{f:\mathrm{hom}(x,y)} \mathrm{isequiv}(f \circ \_) \times \mathrm{isequiv}(\_ \circ f)$$

$$\mathrm{neut}(x) :\equiv \sum_{f:\mathrm{hom}(x,x)} \prod_{y:\mathrm{Ob}} (f \circ \_) = \mathrm{id}_{\mathrm{hom}(y,x)} \times \prod_{y:\mathrm{Ob}} (\_ \circ f) = \mathrm{id}_{\mathrm{hom}(x,y)}$$

Outline

2. Naive Identities

For any object x : Ob we should get

- a particular morphism $i : \hom(x, x)$.
- left and right neutrality: $\lambda_f : i \circ f = f$ and $\rho_f : f \circ i = f$.

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Which in HoTT is written:

$$\prod_{x:\mathrm{Ob}} \sum_{i:\hom(x,x)} \prod_{y:\mathrm{Ob}} \left(\prod\nolimits_{(f:\hom(x,y))} i \circ f = f\right) \times \left(\prod\nolimits_{(f:\hom(x,y))} f \circ i = f\right)$$

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This is not a proposition, we need the propositional truncation.

Naive Identities

$$\mathrm{NaId} :\equiv \prod_{x:\mathrm{Ob}} \|\,\mathrm{neut}(x)\|$$

Outline

1. Wild semicategory
2. Naive Identities
3. (Co)slice approach
4. Harpaz's identities
5. Idempotent Equivalences
6. Comparison

3. (Co)slice approach

Reminder

For an object x in a category C, the slice category C/x (resp. coslice category $x\backslash C$) consists of morphisms $a \to x$ (resp. $x \to a$) as objects and of commutative triangles as morphisms.

$$\begin{array}{ccc} a & \longrightarrow & b \\ & \searrow \quad \swarrow & \\ & x & \end{array} \qquad \left(\text{resp. } \begin{array}{ccc} a & \longrightarrow & b \\ & \nwarrow \quad \nearrow & \\ & x & \end{array} \right)$$

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Remark

In the slice category C/x the identity $\mathrm{id}_x : x \to x$ is terminal.
Dually, id_x is initial in $x\backslash C$.

$\rightsquigarrow$ Use this fact to define the identity structure in a wild semicategory.

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A straightforward way to express the structure is thus to ask for all slices to have a terminal object and all coslices to have an initial object:

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$$\prod_{x:\mathsf{Ob}} \mathsf{hasTerm}(C/x) \times \mathsf{hasInit}(x \backslash C)$$

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A straightforward way to express the structure is thus to ask for all slices to have a terminal object and all coslices to have an initial object:

$$\prod_{x:\mathrm{Ob}} \sum_{t:\mathrm{hom}(\_,x)} \mathrm{isequiv}(t \circ \_) \times \sum_{i:\mathrm{hom}(x,\_)} \mathrm{isequiv}(\_ \circ i)$$

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Issues:

- i and t might not have same (co)domain.
- $\hom(x, x)$ might be empty.

$\rightsquigarrow \mathbb{Z}$ as a semicategory is a counterexample.

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Identity structure using (co)slice approach

$$\mathrm{SliceId} :\equiv \prod_{x:\mathrm{Ob}} \exists_{i:\hom(x,x)} \mathrm{isequiv}(i \circ \_) \times \mathrm{isequiv}(\_ \circ i)$$

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Harpaz[1] gives a similar identity structure but weaker:

Harpaz's Identities

$$\mathsf{HarpazId} :\equiv \prod_{x:\mathsf{Ob}} \exists_{y:\mathsf{Ob}} \mathsf{eqv}(x, y)$$

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Harpaz's Identities

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Which can easily be made into an univalent identity structure:

Univalent Harpaz's Identities

$$\mathsf{uHarpazId} :\equiv \prod_{x:\mathsf{Ob}} \mathsf{isContr}\left(\sum_{y:\mathsf{Ob}} \mathsf{eqv}(x, y)\right)$$

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Question

Can we avoid the use of propositional truncation?

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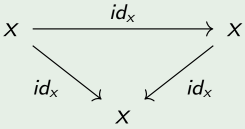
In C/x and $x\backslash C$ the only endomorphism of the identity is the identity itself:



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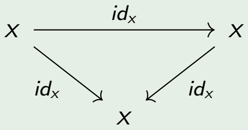

For a morphism f, the property $f \circ f = f$ is called idempotency.

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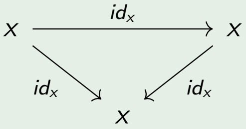

For a morphism f, the property $f \circ f = f$ is called idempotency.

Hence, identity-like morphisms should have this property.

5. Idempotent Equivalences

This kind of identity structure have been studied by Saavera[2] and Kock[3]. Nicolai showed in his paper[4] that idempotency is actually enough:

[2] N. Saavera Rivano, Catégories Tannakiennes (1972)

[3] J. Kock, Elementary remarks on units in monoidal categories (2008)

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Proposition (Kraus, 2021)

The type

$$\mathsf{IdemEqv} :\equiv \prod_{x:\mathsf{Ob}} \sum_{i:\mathsf{eqv}(x,x)} i \circ i = i$$

is a proposition.

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Hence we can take it as identity structure on a wild semicategory.

- No truncation needed.
- We can project out the identity morphism and easily define left and right neutrality (λ and ρ).

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6. Comparison

Theorem

For a given wild semicategory, the four types NaId, IdemEqv, HarpazId, SliceId *are equivalent propositions.*

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Proof.

- The equivalence NaId $\Leftrightarrow$ IdemEqv is straightforward.
- HarpazId $\Rightarrow$ IdemEqv amounts to showing that for an equivalence $x \xrightarrow{f} y$ the morphisms $(f \circ \_)^{-1}(f)$ is an idempotent equivalence.
- IdemEqv $\Rightarrow$ SliceId is the projection.
- SliceId $\Rightarrow$ HarpazId is trivial. □

Conclusion

Overview

Id Structure	Avoids truncation	Captures non-univalent categories
NaId	✗	✓
SliceId	✗	✓
HarpazId	✗	✓
uHarpazId	✓	✗
IdemEqv	✓	✓

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Conjecture

Low levels are enough to produce all the higher coherences for the identity structure.

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Conjecture

Low levels are enough to produce all the higher coherences for the identity structure.

Formalisation in Agda:
See `github.com/jaycech3n/semicategories-with-identities` for details.